
AN EXHAUSTIVE EXPERIMENTAL STUDY OF SYNCHRONIZATION BY FORCING ON ELEMENTARY CELLULAR AUTOMATA

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ABSTRACT. We study a way of coupling two configurations of the same cellular automaton rule for all elementary cellular automata (ECA). We experimentally show that there are only two possible behaviors: either synchronization for all coupling strength, or a phase transition. This transition is shown to belong to the directed percolation universality class, even for a non chaotic rule and for rules with particles.

Introduction

Chaotic systems, with an apparently random behavior, have received a great deal of attention in the last decades. It might be expected that, when adding noise to the system, the behavior becomes even more chaotic. However, in various systems, a transition from chaotic to non-chaotic behavior has been observed when varying a parameter: for some values of the parameter, all trajectories become identical after a while and the system is no more chaotic, since trajectories are independent of initial conditions. Synchronization is made possible by the fact that all instances of the system are subject to the same realization of the noise. This idea of synchronizing systems by identical random perturbation can be traced back to [9] or [11].

The synchronization of simple dynamical systems taking cellular automata (CA) as a model is the subject of a survey in [15], together with other extended dynamical systems. The same authors in [10] describe a stochastic synchronization technique for CA: one considers two configurations initialized independently and randomly. Both follow the same CA rule. To try to synchronize them, one compares both configurations at each time step, cell by cell. If both cells differ, they are made equal with probability p . The set of cells that are made equal, or synchronized, is thus random and changing at each time step. The parameter p controls the strength of the synchronization.

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The goal of this paper is to explore this emergent phenomenon on all ECA (elementary CA, that is, CA with one dimension, two states and two nearest neighbors). Preceding papers studied only one or two rules, chosen for the ease of simulation or for known chaotic properties. We show that one does not need a chaotic CA to observe interesting behavior (in this case, a phase transition) in this setting.

We study by simulation the behavior of all ECA with regard to this synchronization scheme and show that over the 88 different ECA, two behaviors are possible: synchronization even with the slightest synchronization strength, or synchronization only above a certain synchronization strength. For the second class, we study the transition between non synchronization and synchronization when p varies. There is always a phase transition belonging to the universality class of directed percolation. We thus get a new model of directed percolation, with a few variants. Like a few other directed percolation models [6, 10, 12], the limit of the sub-critical regime is neither a single absorbing state, nor a set of fixed points, but a non trivially evolving phase.

Perturbation. The model presented here is a kind of perturbation to the original CA. The general idea behind perturbation is to study the robustness of the system. Real systems are not as regular and defect-free as models, so a good predictive model has to be robust to small perturbation. We here study one kind of perturbation and apply it to all ECA.

Perturbing a system is a first step towards controlling it, indeed, some systems are controlled with small, carefully chosen perturbations (e.g. satellite trajectories). The perturbation studied here is a kind of “self-perturbation”, in the sense that it is a perturbation induced by a CA following the same rule. Such a perturbation, with an external force that is related to the system of interest, might be more relevant than random noise.

Directed percolation. Directed percolation is found in other variations of the CA model. The authors of [2] have studied a continuous model that collapses to deterministic CA dynamics. They show that the observed synchronization transition, on changing the strength of the stochastic coupling between replicas, belongs to the directed percolation universality class. In [12], we presented another way of coupling two configurations of the same CA rule, called coalescence. For some rules, we observed that there is a phase transition between coalescence (the coupling makes both configurations equal) and non coalescence. As predicted by a conjecture from Grassberger [5], the transition belonged to the universality class of directed percolation. The conditions for this conjecture also apply to almost all rules of the present study, and this paper shows that the new model also belongs to this universality class.

The paper is organized as follows. Section 1 gives definitions, notations, and a few remarks. We describe the exhaustive simulation study in Section 2.1, then introduce directed percolation in Section 2.2, and finally check the directed percolation hypothesis and analyze the results in Section 2.3.

1. Definitions and notations

In this section we recall the definition of a CA to fix notations, then define the perturbed CA.

1.1. Usual definition of CA

Definition 1.1. A Cellular Automaton (CA) is a tuple (Q, d, V, δ) where

- Q is the finite set of *states*;
- $d \in \mathbb{N}^*$ is the *dimension*;
- $V = \{v_1, \dots, v_{|V|}\}$, the *neighborhood*, is a finite set of vectors in \mathbb{Z}^d ;
- $\delta : Q^{|V|} \rightarrow Q$ is the *transition rule*;

The *cell space* is $\mathcal{U} := \mathbb{Z}^d$. A *configuration* sets the state of each cell: it is a function $c : \mathcal{U} \rightarrow Q$.

Here is the dynamic: given a configuration c , the next configuration c' is obtained by updating all sites at once by applying δ : $c'(z) := \delta(c(z + v_1), \dots, c(z + v_{|V|}))$. We extend the notation δ by defining $\delta(c) := c'$.

1.2. The Forcing Model

Given a CA, we consider two initial configurations c_0^1 and c_0^2 . Those configurations are random, the state of each cell is drawn independently from the the others, with all states of Q equiprobable. At each time step, each configuration is updated according to δ , then a stochastic synchronization step F_q between both configurations occurs. The event studied is whether both configuration eventually become identical (they are said to have *synchronized*).

F_q consists in, for each cell z independently, doing nothing with probability q , and forcing both configurations to have the same state with probability $1 - q$. When we force both configurations to agree on cell z , the state is chosen randomly uniformly between $c^1(z)$ and $c^2(z)$:

$$\text{For each cell } z \text{ independently, } F_q(c^1(z), c^2(z)) := \begin{cases} c^1(z), c^2(z) & \text{with probability } q \\ c^1(z), c^1(z) & \text{with probability } \frac{1-q}{2} \\ c^2(z), c^2(z) & \text{with probability } \frac{1-q}{2} \end{cases}$$

Combining F_q and δ , we get:

$$c_{t+1}^i(z) := \begin{cases} \delta(c_t^i)(z) & \text{with probability } q & (a) \\ \delta(c_t^1)(z) & \text{with probability } \frac{1-q}{2} & (b) \\ \delta(c_t^2)(z) & \text{with probability } \frac{1-q}{2} & (c) \end{cases}$$

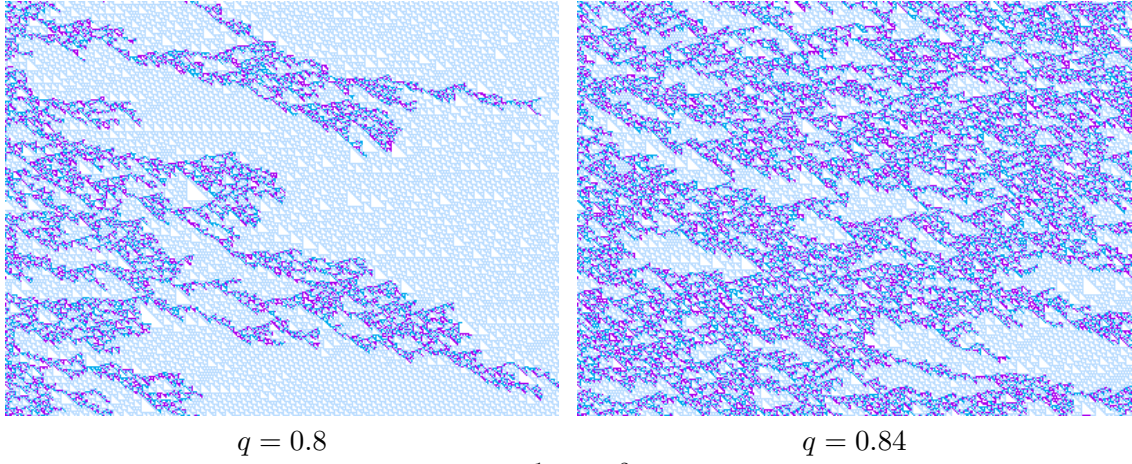
For each cell z independently, the same choice among (a), (b) and (c) is made for both configurations. An example is given on Figure 1.

The probability q is a parameter of the model. The case $q = 1$ corresponds to the unperturbed CA, or two independently evolving configurations, while the case $q = 0$ imply total synchronization just after the first step.

Note that if both states are equal, the forcing has no effect. Since the decision of forcing or not is independent for each cell, we can equivalently say that we try to force only if both cells are different. This is the presentation chosen in [10].

Proposition 1.2 (the case of strong coupling). *On finite configurations of size n , if $q \leq \frac{1}{|V|}$ then synchronization occurs in $O(n)$ expected time.*

Proof. The density of disagreement cells is, on average, multiplied by q each time F_q is applied. It is multiplied by at most $|V|$ when applying δ , since a disagreement cell can make only its neighbors become disagreement cells. If $|V|q < 1$, the expectancy of this density is thus exponentially decreasing. ■



Time goes from left to right. Configurations c^1 and c^2 are superimposed. Agreeing cells (i.e. cells having the same state in both configuration) are plotted light, disagreeing cells are plotted dark. One can see typical patterns of the original rule 110 in the light zones.

Figure 1: Forcing model applied on rule 110.

The bound on q of proposition 1.2 is loose:

- not many rules make disagreement spread at the speed of light in all configurations,
- disagreement sites are not isolated, precisely because they spread.

Link to Another Model. In [13], the authors study on finite configurations what they call self-synchronization, i.e. they have only one configuration that they try to “synchronize” with itself, which means reaching a stable configuration. They do it by setting $c_t^2 := c_{t-1}^1$, i.e. synchronizing the configuration with the configuration at the previous time step. This means that at each time step t , for each cell independently, the cell is reset to its state at step $t - 1$ with probability $\frac{1-q}{2}$. This is equivalent to updating the cell with probability $\frac{q+1}{2}$ and doing nothing otherwise. The latter model, which updates only some cells at each time step, has been studied extensively both experimentally and analytically in [4].

2. Experimental study

In this section, we systematically study (in the forcing model) the ECA, that is, CA with $Q = \{0, 1\}$, $d = 1$, $V = \{-1, 0, 1\}$. We show that there are only 2 possible behaviors: synchronization even with the slightest forcing strength, and phase transition when q varies.

2.1. Classification of ECA in the forcing model

Here is the protocol of our experiments. We call *run* the temporal evolution of a CA when all parameters (rule, size n , probability q and two initial configurations) are chosen. We stop the run when both configurations are synchronized, or when a predefined maximum running time has been reached.

Let us describe the parameters we used.

- (1) Number of cells n . The main points when choosing n is to check that the results do not depend on a particular choice of n , in particular that n is big enough. Some authors (like [14]) suggest that small is enough ($n = 30$), others (like [3]) state the opposite, and we follow the latter. A similar problem studied in [4] shows a stable behavior for $n \geq 200$. We set $n = 2\,000$ and check that the results do not change for $n = 500$.
- (2) Number of computation steps. To measure the asymptotic density of disagreeing cells ρ , we let the automaton run for 200 000 steps, then measure the density averaged over 10 000 steps.

According to directed percolation theory, near a phase transition, the automaton can take an arbitrarily long time before settling down to the asymptotic density. So, for a few choices of q , those parameters are not sufficient to measure the true asymptotic density. However, they are big enough to *detect* that there is a transition point, and then study more precisely what happens there.

- (3) q . We try to sample the entire range. For each of the 88 rules, we do 999 runs: one for each value of q ranging from 0.001 to 0.999.

One might want to average over many runs. To show that we do not need to, we plot ρ versus q . The smoothness of the resulting curve (Figure 2) shows that the variance between runs is low.

The random seed for deciding which cells to update at each step is distinct for each value of q .

- (4) The initial configurations are random (each cell is in state 0 with probability 0.5, independently from the other cells) and distinct for each value of q .

When we apply this protocol to all ECA, there are only two different situations occurring. A typical plot of each case is on Figure 2.

- 68 ECA have a trivial behavior: both configurations always synchronize within the given time, for all values of q .
- 20 ECA exhibit a phase transition. For some q_c (depending on the rule):
 - If $q < q_c$, both configurations rapidly synchronize.
 - If $q > q_c$, the density of disagreeing cells settles to non zero value for a long time.

We deal with finite configurations and can thus be subject to finite size effects. One effect of notable importance is the following. Take a rule with a phase transition, there is an update rate q for which the rule is synchronizing, i.e. the pair of configurations reaches total agreement in polynomial time. In the non synchronizing regime, with low probability, the outcome of the random bits determining which cells to force can make the CA simulate the synchronizing regime for a fixed number of steps. So, if a CA can synchronize for a given q , it can synchronize for any $0 < q < 1$. The true asymptotic regime is thus always synchronization.

In other words, when the density settles to non zero value (case $q > q_c$), it still fluctuates randomly around this value. Fluctuations eventually make the density touch 0, which is absorbing.

However, these fluctuations become smaller as n increases, and the limit when $n \rightarrow \infty$ should be that the density does not reach 0 anymore. Moreover, this first experiment is used to detect rules that have an interesting behaviour, and all rules of the second class will be checked in detail in Section 2.3.

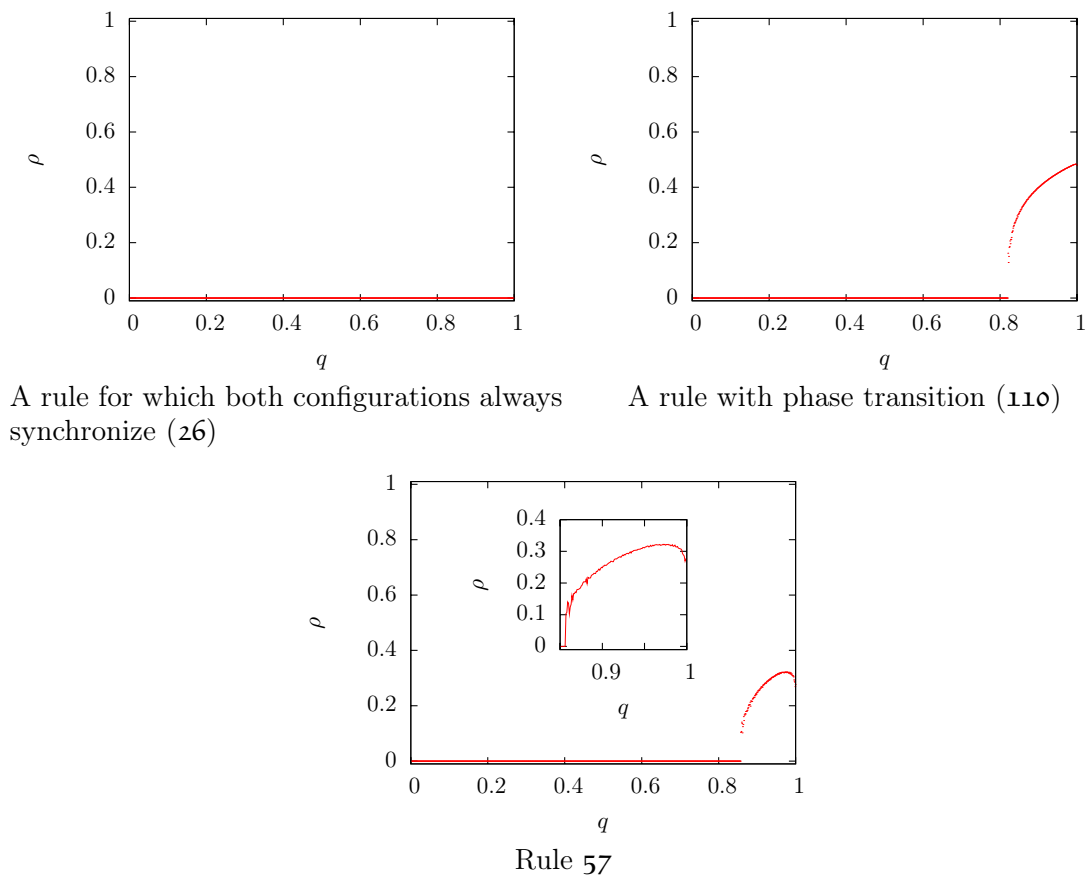


Figure 2: Asymptotic density of disagreeing cells ρ versus q .

2.2. Phase transition and directed percolation

This section recalls the minimal background about directed percolation.

2.2.1. *Phase transition.* A phase transition is an abrupt change in macroscopic properties of a system with only a small change of a control parameter, say T , around a critical value T_c . This paper is concerned only with second order phase transitions, or continuous phase transitions, which can be characterized by *critical exponents*. If one let the parameter T vary near the phase transition (occurring at $T = T_c$), all other variables being fixed, a measurable quantity C has a power law behavior $C \propto |T - T_c|^\beta$ at least on one side of T_c . Several exponents are defined, depending on the quantity measured.

Remarkably, many systems with no *a priori* relation turn out to have the same critical exponents. A *universality class* is defined as all the systems having the same set of critical exponents.

2.2.2. *A conjecture on damage spreading.* Chaos theory deals with the sensitivity to initial condition of *deterministic* systems. To also study the influence of small perturbations on stochastic systems, the authors of [8] introduced *damage spreading*. In this model, two copies

of a stochastic model are run in parallel with the same source of random bits, starting from different initial configurations (often they are set to differ in exactly one site).

One measures the temporal evolution of the proportion of differing sites, called the Hamming distance. If this goes to zero, i.e. if both copies become identical, the initial “damage” has “healed”, otherwise the damage is said to spread.

There is a conjecture by [5] stating that, if a transition occurs between healing and spreading in a stochastic spin model, the universality class of this phase transition is always the same, namely the one of *directed percolation*, which is presented in the next paragraph.

There are some conditions for this conjecture:

- (1) Only short range interactions in time and space,
- (2) translational invariance,
- (3) non vanishing probability for a site to become healed locally,
- (4) the transition does not coincide with another phase transition.

Point 2 is easily fulfilled for CA. Point 3 is a direct consequence of the definition of the forcing model. About point 4 (no simultaneous transition), we know of no other transition. We discuss point 1 in Section 2.3.3.

2.2.3. The Model of Directed Percolation. A more detailed introduction to directed percolation can be found in [4] (note that this paper cites a different conjecture of Grassberger than the one we deal with). A survey of directed percolation is contained in [7], which also covers *damage spreading*.

Isotropic percolation was first defined when studying propagation of a fluid through a porous medium. It has been mathematically modelled as an infinite square grid where each site has the four nearest sites as neighbors. Each bond between two neighbors can be open (letting the fluid go through) with probability p or closed with probability $1 - p$, independently of all other bonds.

The question is whether the fluid inserted at one point will pass through the medium, i.e. whether this point is part of an infinite network of sites connected by open bonds.

Directed percolation appears when one adds gravity to the model, i.e. when the fluid is only allowed to travel in one direction (Figure 3). Static 2D directed percolation can also be seen as a 1D dynamical model where some sites are “active” (where active can mean wet, infected, etc.). An active cell can stay active or die (become inactive), and make its neighbors active. Depending on the probabilities of these possibilities, active regions spread or disappear. Cells can only have an influence on the future states of their neighbors, thus the *directed* percolation.

The macroscopic quantity measured is the density of active states as a function of p and time, $\rho(p, t)$. It is zero in one phase and non-zero in the other. There exists a critical probability p_c which is the limit between two phases:

- For $p < p_c$, the asymptotic density $\rho(p, \infty)$ is 0;
- for $p > p_c$ we have a power law $\rho(p, \infty) \propto (p - p_c)^\beta$;
- for $p = p_c$ the density goes to 0 as $\rho(p_c, t) \propto t^{-\delta}$.

2.3. Directed percolation in the forcing model

In our model, the active sites are the cells where the configurations disagree. Asymptotic density of such sites is written $\rho(q)$. The pairs of configurations where all cells agree

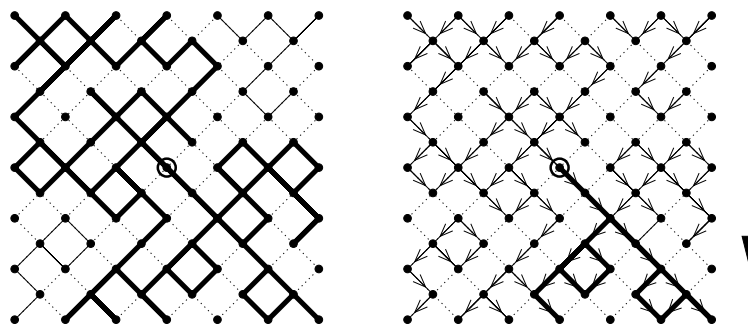


Figure 3: Isotropic (left) and directed (right) percolation. Figure reprinted from [7].

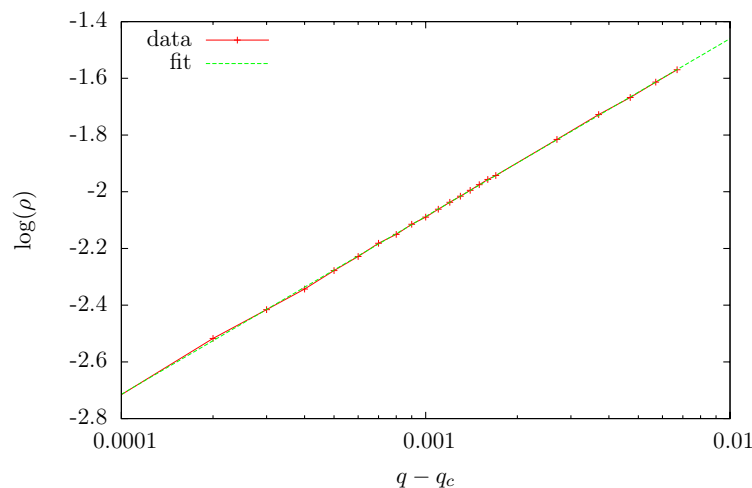


Figure 4: Measuring β for rule **110**.

constitute the absorbing set. Percolation transition (synchronization or not) appears when varying q , see Figure 1. (Note that there is no direct relation between q and p , because of the underlying CA dynamic.) The aim is thus to identify β assuming that

$$\rho(q) \propto |q - q_c|^\beta \tag{2.1}$$

for some q_c . Like many authors [4, 5], we will focus on β and consider it as sufficient to test directed percolation.

2.3.1. *Measure of β .* Two methods to measure β were compared in [12], we use the following one. We plot $\log \rho(q)$ versus $\log(q - q_c)$ for values of q near q_c and adjust q_c to get a straight line. Once q_c is fixed, we fit a straight line, the slope of which is an estimator of β . See Figure 4 and Equation 2.1. It is important to do the fit against $\log \rho$ (and not ρ), so that all errors get the same weight when fitting a line on the log-log plot.

The protocol is only semi-automatic. We try increasing values of n between 10 000 and 1 000 000 to get a reasonably smooth line on the density versus time plot. All other parameters being set, different values of n give fluctuations around the *same* asymptotic

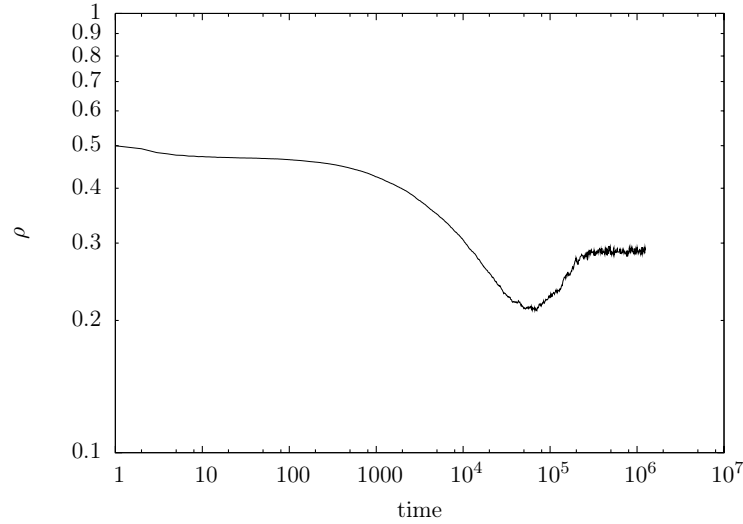


Figure 5: ρ versus time for rule 58 and $q = 0.9999$ (log log scale).

density, and larger n yields smaller fluctuations and thus greater precision. Also, this allows us to check that the number of steps required for the density versus time plot to become horizontal is not affected by n .

We then visually check that the density has reached a steady state, then average the density over at least half a decade. This yields one measure point. We repeat this process for several values of q near q_c .

Note the misleading diagram of rule 58 (Figure 5): running the experiment only for 50 000 time steps would lead one to conclude that the density will reach zero.

2.3.2. Results. The fit gives the ranges of Table 1, taking into account uncertainty about q_c and which points to keep for the fit. Experimental value for β measured on other systems is 0.276. As expected, all models undergoing a phase transition (that is, in our case, all models not always synchronizing) seem to belong to the universality class of directed percolation.

Rule 110 has been measured with higher precision than the other rules to check the influence of particles, discussed in Section 2.3.3.

Thresholds for rules 58 and 62 are very close to 1 and too sparse sampling of the values of q could miss their transition. For those rules, a tiny synchronization strength (roughly one percent) is already enough for both configurations to synchronize, but there is still a strength at which they do not synchronize.

A few rules have very close thresholds and this is no coincidence: there is a way to relate their dynamics.

- **18** and **146**. The only difference between those rules is on the local configuration 111: $\delta(1, 1, 1) = 0$ for **18** and $\delta(1, 1, 1) = 1$ for **146**. But the only way to have a pattern 1^k (with $k \geq 3$) under the dynamic of **146** is to have the pattern 1^{k+2} on the previous configuration. So, provided there is at least one 0 in the initial configuration, such patterns rapidly disappear and the dynamics of **18** and **146** are then identical.

rule	q_c	β
9	0.9630(2)	0.306 ±0.031
18	0.8092(2)	0.285 ±0.021
22	0.7727(2)	0.268 ±0.018
25	0.9570(2)	0.288 ±0.016
30	0.7935(1)	0.269 ±0.016
41	0.7954(1)	0.277 ±0.013
45	0.7946(1)	0.275 ±0.011
54	0.8387(2)	0.283 ±0.012
57	0.8546(1)	0.295 ±0.026
58	0.9968(2)	0.26 ±0.03
60	0.8094(1)	0.27 ±0.015
62	0.9854(2)	0.291 ±0.027
90	0.8094(2)	0.263 ±0.022
105	0.6789(2)	0.268 ±0.01
106	0.8498(1)	0.275 ±0.009
110	0.81930(1)	0.272 ±0.005
122	0.7850(1)	0.274 ±0.011
126	0.7892(2)	0.269 ±0.011
146	0.8094(2)	0.259 ±0.021
150	0.6789(2)	0.265 ±0.013

Numbers in parenthesis give the precision:
 “0.9630(2)” means $q \in [0.9628; 0.9632]$.

Table 1: q_c and β for all ECA undergoing a phase transition.

- **60** and **90**. **90** means “xor between my left and my right neighbors”. **60** means “xor between me and my left neighbor”. Let us consider a space-time diagram $\{c_t(z) \mid t \in \mathbb{N}, z \in \mathcal{U}\}$ obeying rule **90**:

$$\forall z \quad \forall t \quad c_{t+1}(z) = \delta(c_t(z-1), c_t(z), c_t(z+1)) = c_t(z-1) \oplus c_t(z+1)$$

If we extract the space-time diagram $\{c'_t(z) := c_t(2z - t) \mid t \in \mathbb{N}, z \in \mathcal{U}\}$, it obeys rule **60**:

$$\begin{aligned} \forall z \quad \forall t \quad c'_{t+1}(z) &= c_{t+1}(2z - t - 1) \\ &= c_t(2z - t - 2) \oplus c_t(2z - t) \\ &= c'_t(z - 1) \oplus c'_t(z) \end{aligned}$$

Thus, **90** simulates two half-size configurations of **60**.

- **105** and **150**. **150** means “xor of the states of the neighbors” while **105** means “compute the output of **150** and take the opposite state”. Thus, the agreement status of on cell only depends on the agreement status of its neighbors in the previous configurations (not on its actual state in both configurations). Which allows us to conclude that, if we run both rules on the same pair of initial configurations, exactly the same cells at each time step will disagree.

2.3.3. *The case of rule 57.* In previous papers, only chaotic rules have been the subject of interest for studying synchronization and phase transition in the forcing model. But let us consider rule **57** which, when unperturbed, converges quickly to a period 2 orbit (in this case a checkerboard). It is thus not chaotic. Nonetheless, if one adds forcing, a

phase transition occurs. This shows that this phase transition does not require the CA to be chaotic. However, all ECA in class 3 or 4 of Wolfram classification (i.e. “chaotic” or “complex” ECA) undergo a phase transition.

This rule is of further interest because it has particles (in the forcing model) and thus long range correlations, as seen on Figure 6. Rule 110 is also known to have particles (see Figure 1) and thus exactly the same kind of long range correlations. We have shown that both rules undergo a phase transition of the directed percolation class. This shows that point 1 of Grassberger’s conjecture (page 256), while useful for discarding models that do not belong to this class, might be too restrictive.

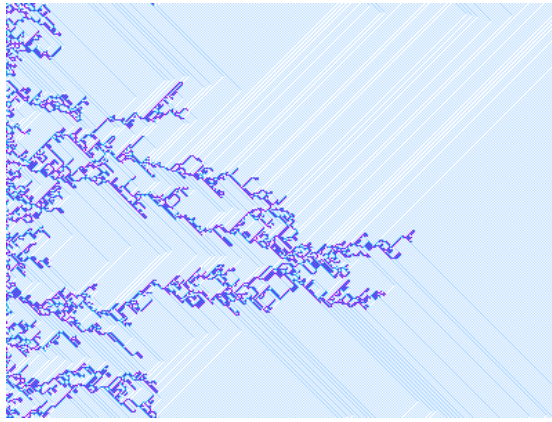


Figure 6: A Space-time diagram of rule 57 in the forcing model.

The last insight given by this rule is the following. On Figure 2.c, one sees that for q between 0.98 and 0.999, more forcing (q closer to 0) means higher asymptotic density. In other words, increasing the number of cells we force to be equal at each time step makes the density of disagreeing cells higher. Note that applying the forcing step F_q is equivalent to applying two forcing steps $F_{\sqrt{q}}$. In this case, applying the second one would raise ρ .

3. Conclusion and perspectives

We have studied a way to perturb a cellular automaton, with a perturbation related to the original rule. For 68 ECA rules, the slightest perturbation allows to synchronize two random initial configurations. For the 20 remaining ones, there is a threshold on the coupling strength.

When studying the behavior close to the threshold, we confirm the results of the first experiment and show that there is a phase transition belonging to the universality class of directed percolation.

Experiments have to be run on finite configurations, but we checked that the number of cells n does not influence the outcome, increasing n only increases accuracy. There is evidence that the only finite size effect (the fact that zero density is absorbing) occurs only after an exponential time, and thus does no harm here.

Therefore, we expect the theoretical model to behave like the experiments, i.e. to undergo a phase transition with an exponent of $\beta \simeq 0.276$ (note that about the 1D directed

percolation, the literature have precise experimental values for β but no analytical derivation).

There are certainly some rules that can be analytically studied and be proven of undergoing a phase transition. Simulation between directed percolation models is a strategy for this kind of result.

Other open questions remain. An obvious generalization is to test in which proportion this phenomenon occurs in CA with more states, more dimensions or more neighbors.

Also, in this model, there is a symmetry between both configurations (they are treated equally). In a context where the aim is to control the system, it would be natural to study a “master/slave” setting: when the random outcome tells to make two cells equal, the state would always be copied from the first configuration to the second.

Finally, one could imagine relevant ways of coupling more than two instances of the CA. But it would be even more interesting to mix this coupling scheme with the coupling studied in [12] and see how their respective effects combine.

Links. Space-time diagrams for all ECA can be found on <http://www.rouquier.org/jb/recherche/eca>. Source code used for the simulations is at <http://cimula.sf.net>.

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