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TILINGS AND MODEL THEORY

ALEXIS BALLIER [1] AND EMMANUEL JEANDEL [1]

E-mail address: Alexis.Ballier@lif.univ-mrs.fr

E-mail address: Emmanuel.Jeandel@lif.univ-mrs.fr

[1] Laboratoire d'informatique fondamentale de Marseille (LIF), Aix-Marseille Université, CNRS, 39 rue Joliot-Curie, 13 453 Marseille Cedex 13, France

ABSTRACT. In this paper we emphasize the links between model theory and tilings. More precisely, after giving the definitions of what tilings are, we give a natural way to have an interpretation of the tiling rules in first order logics. This opens the way to map some model theoretical properties onto some properties of sets of tilings, or tilings themselves.

1. Introduction

Tilings are a basic and intuitive way to express geometrical constraints; they happened to be of broad interest in computer science since Berger proved the undecidability of the domino problem [2] by showing that they can embed, despite being static objects, some kind of computation. This also was the first step in the links between logics and tilings as they helped to prove the undecidability of some classes for formulae [5, 14, 12, 13]. Some more links have then been discovered by Makowsky that used previous constructions of aperiodic tilesets to show the existence of a complete, finetely axiomatizable and superstable theory [9]. Some recent results by Oger generalize this approach to more abstract definitions of tilings and proves some nice equivalences between model theory and this generalized definition [10].

In this paper we will give details of constructions used to translate tilings and tileset properties into model theoretic ones. Section 2 will be devoted to the proper definitions of tilings and tilesets; We will then translate these definitions into first order formulae in Section 3. Finally in Section 4 we shall present the equivalence results that can be obtained by this translation.

Most of these results are already present in [9, 11]. However we hope that this paper will offer a new look at these results.

The major part of this paper is devoted to tilings of the plane $\mathbb{Z}^2$. However, we may define similar theories for tilings of other spaces such as $\mathbb{Z}^3$ or any Cayley graph. The article [10] in particular deals with tilings of $\mathbb{R}^n$ by polytopes.

2. Tilings

Several definitions of discrete tilings can be found in the litterature, but are equivalent for many purposes [3]. We will focus here on the definition by forbidden patterns.

First we have to define the space we are going to tile: we want to assign a state taken in a finite set Q to each cell of the discrete plane $\mathbb{Z}^2$. Q may be seen as a set of colors, or a set of states. Therefore, we define the set of configurations as the functions from $\mathbb{Z}^2$ to Q:

Definition 2.1. The set of configurations is $Q^{\mathbb{Z}^2}$.

The patterns are nothing but a configuration restricted to a finite domain; that is, considering a finite subset D of $\mathbb{Z}^2$, a pattern is a function from D to Q.

Definition 2.2. A pattern defined on a finite subset D of $\mathbb{Z}^2$ is an element of Q^D.

Informally a tileset represents geometric constraints imposed to the configurations, that is how the states in the cells of the plane are constrained by their neighborhood and how they constrain it. Formally we will define a valid tiling as a configuration that contain no forbidden pattern:

Definition 2.3. A tileset is defined by a finite set of forbidden patterns $\mathcal{F}_\tau$.

A configuration c contains a pattern P defined on D (or equivalently P appears in c) if there exists $x \in \mathbb{Z}^2$ such that:

$$\forall y \in D, c(x+y) = P(y)$$

A configuration is said to be a valid tiling by τ if it contains no pattern in $\mathcal{F}_\tau$.

The so-called domino problem [2] is to know given a tileset whether it generates a valid tiling. The problem has been proven undecidable by Berger in [2].

We will now define a preorder $\preceq$ on configurations that focuses on patterns contained in them. This preorder has been defined in [4, 1], however references to the concept can be found as early as [11]:

Definition 2.4 (The pre-order $\preceq$)**.** Let x, y be two configurations, we say that $x \preceq y$ if any pattern that appears in x also appears in y.

This induces the notion of local isomorphism between two configurations:

Definition 2.5 (Local isomorphism)**.** Two configurations x and y are said to be locally isomorphic if $x \preceq y$ and $y \preceq x$. That is x and y contain the same patterns. We denote it by $x \approx y$.

Two configurations that are equal up to shift are locally isomorphic but the converse is not always true: there exists configurations that are locally isomorphic but one is not a shifted form of the other.

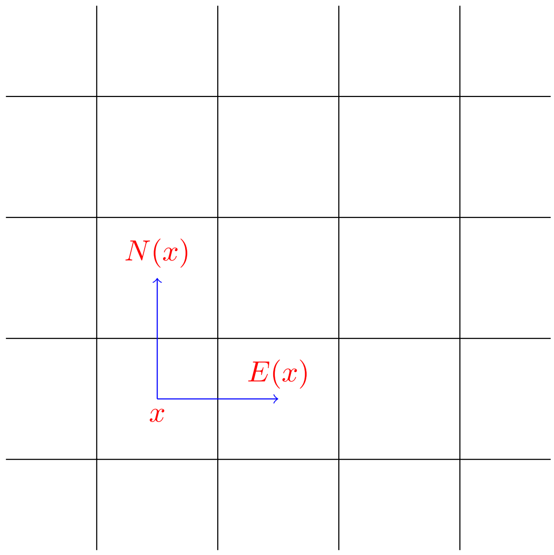

Figure 1: The model we would like to obtain

3. From tilesets to model theory

In this section we translate the definitions given in Section 2 into first order formulae on some given language. This translation maps some properties of tilings onto some other properties of first order logics.

Such a correspondence between tilesets and first order logic has already been defined [11, 9] to show an example of finitely axiomatizable and superstable theory. A similar approach (see 3.4) has been used to prove the undecidability of certain classes of formulae [13, 14, 12, 5].

3.1. Axiomatizing the plane

The ideal model we would like to obtain is the plane $\mathbb{Z}^2$ like depicted in Figure 1. The natural way to define cells on the plane $\mathbb{Z}^2$ is to consider them as variables and the adjacency relations between them as functions that allow us to move north, south, east or west from a given cell:

Definition 3.1. We consider the language with the unary functions for movements on the plane: $\mathcal{L}_0$ is a set of unary functions : $\mathcal{L}_0 = \{N, S, E, W\}$.

And the corresponding axioms of the plane $\mathbb{Z}^2$:

- $\forall x, N(S(x)) = S(N(x)) = E(W(x)) = W(E(x)) = x$
- $\forall x, N(E(x)) = E(N(x))$

These formulae tends to axiomatize $\mathbb{Z}^2$ as a Cayley graph with two generators, the first formula axiomatizing the invertibility of the movements and the second the commutativity. However, these axioms are not sufficient, as we will see in the following sections.

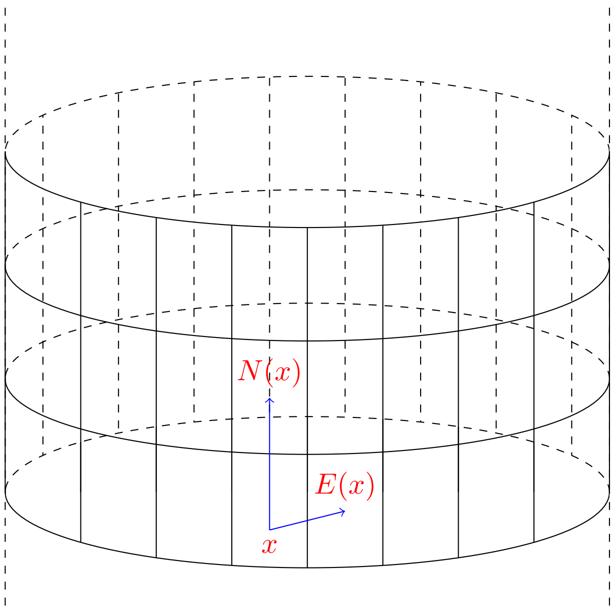

Figure 2: A cylindric model

3.1.1. *Non standard models.* With the axioms of the plane from the previous sections it is still possible to obtain some weird models. First, they also axiomatize some finite models like $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$, or some cylindric models like $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}$ (like e.g., in Figure 2).

This problem can be dealt with by adding more axioms : For any i and j we may add the axiom $\forall x, E^i N^j(x) \neq x$. The main problem is then that the number of axioms is not finite, so that (we can prove that) the theory we obtain is not finitely axiomatisable anymore. However in most cases, the presence of these models is not a problem as we can "unfold" them into a plane (see e.g proof of lemma 4.2).

3.1.2. *Connectedness.* The main problem however, which cannot be avoided, is that there is no way to ensure that all models of our theory are connected : A model is said to be connected if any two points can be connected using the N, S, E, W functions. An example of a disconnected model of our theory is depicted on Figure 3. These disconnected models cannot be avoided. This is e.g., a consequence of the Löwenheim-Skolem theorem (There exist models of our theory of arbitrary infinite cardinals, these models cannot be connected if they are not countable) or more simply can be proven by a simple compactness argument: Consider a theory T that axiomatises the plane $\mathbb{Z}^2$. Add two constants c, d and the formulae ϕ_n that express that the points c and d are at distance at least n. Consider the theory $T' = T \cup \{\phi_n \mid n \in \mathbb{N}\}$. Any arbitary finite part of T' admits $\mathbb{Z}^2$ as a model (choose two points c and d arbitrary far) so that T' itself has a model by compactness. Such a model cannot be connected.

This proof also hints to a way to partially solve the problem. Consider formulae $\phi_n(x, y)$ that express that the points x and y are at distance at most n. Now consider the collection $p(x, y) = \{\phi_n(x, y), n \in \mathbb{N}\}$. $p(x, y)$ is a type, that is we can find for every finite part $q(x, y)$ of $p(x, y)$ some points c and d in any model so that $q(c, d)$ is true. Now we are interested

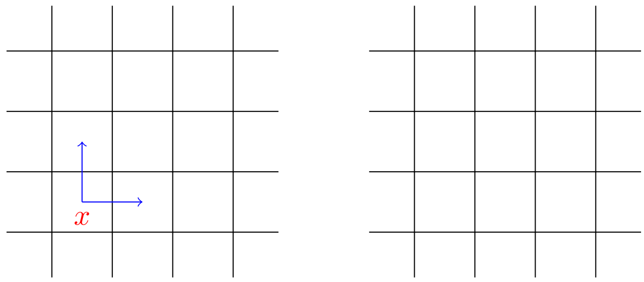

Figure 3: An example of disconnected model

in those models where p is not satisfied, that is in models where there do not exists c and d such that $p(c, d)$ is true. We say that such a model *omits* p

A part of model theory is devoted to the study of models omitting types. As an example, the omitting type theorem states that given a theory T any reasonable type can be omitted. However, most of the classical results in model theory will not work in this context, as e.g. the compactness theorem.

3.2. Encoding configurations

Now that we have some kind of axioms for the plane $\mathbb{Z}^2$, we may define what a configuration is. We defined a configuration as an application from $\mathbb{Z}^2$ to a finite set of states Q. We can code the states of the cells in our theory by unary predicates : we take one predicate Q_i for each state. The only thing we need to ensure is that each cell has exactly one state:

Definition 3.2. New language:

$$\mathcal{L}_Q = \mathcal{L}_0 \cup \{Q_1, \ldots, Q_n\}$$

New axioms:

$$A : \forall x, \bigvee_i Q_i(x)$$

$$B : \forall x, \bigwedge_{j \neq i} (Q_i(x) \Rightarrow \neg Q_j(x))$$

We can also reduce the number of predicates by coding the states in binary form: for example, with 4 predicates, we can code up to 16 states.

3.3. The theory of a tileset

Following our definitions of tilesets in Section 2, all what we need to do in order to encode them in first order logic is to write formulae that express ”some specific pattern never appears”. It can be done in the following way : Given a pattern P of domain D, any point p in D can be represented by a function that is a composition of the functions N, S, E, W. We can then write formulae that express that P appears at a point x:

Definition 3.3. A formula to express that a pattern P defined on D appears at point x
$\varphi_P(x) := \bigwedge_{(i,j)\in D} P(i, j)(E^i(N^j(x)))$

As an example, the formula $\varphi = Q_1(E(x)) \wedge Q_2(x)$ expresses that x is of color 2 and its east neighbour is of color 1.

Then the formula $\forall x, \neg\varphi_P(x)$ axiomatizes that P never appears.

Definition 3.4. The theory T_τ of a tileset τ is the theory over the langage $\mathcal{L}_Q$ that contains all previous formulae and the formula $\forall x, \neg\varphi_P(x)$ for each forbidden pattern P. If the set of forbidden pattern $\mathcal{F}_\tau$ is finite, this theory is finitely axiomatisable.

3.4. Other languages

Before proceeding to the results, we give in this section various other languages in which to express tilings.

Another way to represent tilings is with a single unary function s (that intuitively denotes the successor of an integer) and with binary predicates P_i. $P_i(x, y)$ means that the state in the cell (x, y) is i. A structure is then over $\mathbb{Z}$ rather than $\mathbb{Z}^2$.

It is easy to represent forbidden patterns in this language. As an example, the formula $\phi = \forall x, y, \neg(P_1(x, y) \wedge P_2(s(x), y))$ means that there cannot be a cell in state 1 at the left of a cell in state 2.

Now suppose that the set of forbidden patterns has some particular form, that is constraints only concern adjacent cells. We now have a set of horizontal constraints H ($(i, j) \in H$ if a cell in state i cannot be at the left of a cell in state j) and vertical constraints V.

Now, the constraints can be written in the following way:

$$\phi = \forall x \forall y \bigwedge_{(i,j)\in H} (P_i(x, y) \Rightarrow \neg P_j(s(x), y)) \wedge \bigwedge_{(i,j)\in V} (P_i(x, y) \Rightarrow \neg P_j(x, s(y)))$$

This can be rewritten (by a slight change of variables in the second part of the formula):

$$\forall x \forall y \bigwedge_{(i,j)\in H} (P_i(x, y) \Rightarrow \neg P_j(s(x), y)) \wedge \bigwedge_{(i,j)\in V} (P_i(y, x) \Rightarrow \neg P_j(y, s(x)))$$

Now by a straightforward application of the skolemization process, we can replace the function s by a quantifier :

$$\forall x \exists x' \forall y \bigwedge_{(i,j)\in H} (P_i(x, y) \Rightarrow \neg P_j(x', y)) \wedge \bigwedge_{(i,j)\in V} (P_i(y, x) \Rightarrow \neg P_j(y, x'))$$

We then obtain a new formula ϕ such that ϕ has a model if and only if there exists a tiling of the plane by the tileset. The proof proceeds as in lemma 4.2 below. Note that the unfolding gives us only a tiling of a quarter of the plane. But it is known that a tileset can tile the entire plane if and only if it can tile a quarter of the plane.

The new formula ϕ is a formula with only three quantifiers $\forall\exists\forall$ and which contains only binary predicates. Thus we actually have proven that the class of formulae $[\forall\exists\forall, (0, \omega)]$ is undecidable. This is the core of the works by Wang, Kahr, Büchi about decidability of class of formulae. We then can deduce by an intricate transformation that the Kahr class $[\forall\exists\forall, (\omega, 1)]$ (one binary predicate, a finite number of unary predicates) is also undecidable.

See [14, 12, 13] for more details. The encoding also has another property : The formula ϕ has a finite model if and only if there exists a periodic tiling of the plane by the tileset. This actually proves that the class $[\forall\exists\forall, (0, \omega)]$ is a *conservative reduction class*. See [5] for more details.

4. Translating tilesets and tilings properties in model theoretical ones

We now show the links between those two different approaches.

Lemma 4.1. *A configuration can be seen as a structure over $\mathcal{L}_Q$. A tiling by τ can be seen as a model of T_τ.*

This lemma is a consequence of the definitions we have taken, any configuration is a structure over $\mathcal{L}_Q$ and the construction of T_τ was done in order to forbid patterns that are forbidden by τ, thus a tiling by τ is a model of T_τ.

Lemma 4.2. *T_τ is consistent if and only if τ can tile the plane.*

Proof. It is obvious (by lemma 4.1) that if τ can tile the plane, then T_τ is consistent: A tiling provides a model of T_τ.

Now suppose that T_τ has a model M. We will "unfold" M starting from a point x in it by applying the functions N, S, E, W that will give us any point in $\mathbb{Z}^2$. We can define a configuration c, such that $c(0,0)$ has the "state" of x, and $c(i,j)$ has the "state" of $E^i(N^j((x)))$. This configuration is a tiling : As M is a model of T_τ, no forbidden pattern can appear. Therefore, from any model of T_τ, we can obtain a tiling of the plane by τ, which finishes the proof. ∎

Remark 4.3. T_τ has a model if and only if T_τ has an infinite model.

We can force all models to be infinite by adding (infinitely many) axioms that will ensure this property. The theory may however not be finitely axiomatisable anymore.

Note however that if a tileset does not admit any periodic tiling, no finite models can appear. Moreover, if a tileset does not admit any tiling with at least one direction of periodicity, then all models are only union of copies of $\mathbb{Z}^2$. That is, no degenerate torus or cylinder may appear.

Lemma 4.4. *T_τ has a finite model if and only if τ can tile periodically the plane.*

Proof. Consider a periodic tiling of period p, we "fold" it into $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and obtain a model of T_τ since the cell at position $(x+p, y)$ will have the same state as the one at (x, y) or $(x, y+p)$.

If we have a finite model, we unfold it the same way as in Lemma 4.2. It is easy to see that we obtain this way a periodic tiling. ∎

Most of these results can be generalized to tilings of $\mathbb{R}^2$ using "patches" as tiles and we still get the same translation from tileset and tilings into model theory [10].

4.1. Isomorphism

One of the first properties of models of a given theory one has to study is the isomorphism of models. The translation of this property as properties of tilings is quite straightforward:

Lemma 4.5. *Two configurations are equal up to shift if and only if they are isomorphic as structures on the language $\mathcal{L}_Q$.*

Proof. $\Rightarrow$: Let x, y be two configurations equal up to shift and σ be a shift of vector (i, j) such that $x = \sigma(y)$. Then σ is an isomorphism from x to y.

$\Leftarrow$: Let Θ be the isomorphism and a and b two points of x and y such that $\Theta(a) = b$. Then $E^i N^j(a)$ has the same state as $E^i N^j(b)$, as the predicate $P_q(E^i N^j(x))$ has the same value in a and b since Θ is an isomorphism. ∎

4.2. Elementary equivalence

Another model theoretic property that translates to tilings is the elementary equivalence. We recall that two structures are elementary equivalent if and only if they satisfy the same formulae (that is have the same theory)

Lemma 4.6 ([11, 10])**.** *Two configurations x and y are locally isomorphic if and only if they are elementary equivalent as structures over $\mathcal{L}_Q$.*

Proof. We will consider for the proof x and y as structures over the language without functions, i.e., we replace in $\mathcal{L}_Q$ the functions N, S, E, W by functional predicates N', S', E', W', that is $N'(x, y) \Leftrightarrow N(x) = y$.

$\Leftarrow$: One can express the apparition of the pattern M by a first order formula like in Definition 3.3: $\exists x, \varphi_M(x)$. Therefore, as any formula valid in one structure is valid in the other one, any pattern that appears in one tiling appears in the other one. This proves that if the structures are elementary equivalent then the tilings are locally isomorphic.

$\Rightarrow$: This proof is rather technical and is given in [10] using Hanf locality lemma [6](lemma 2.3). Hanf locality lemma states that for two structures, if the spheres (using the relational distance) all contain finitely many points (what is always true in our case), and if both stuctures have either the same finite number of different spheres or both have an infinite number, then the two structures are elementary equivalent. Hanf locality lemma can be proved using a back and forth method, or an Ehrenfeucht-Fraïssé game.

In our case, the spheres represent the patterns: Consider a point x and all the points at relational distance at most n, since our language contains only binary predicates and that they represent the functions N, S, E, W, the relational distance is nothing but the L_1 distance (or Manhattan distance or also Taxicab Metric) on $\mathbb{Z}^2$. Therefore the sphere at point x of radius n is the pattern defined on $B_1(x, n)$.

Both configurations x and y have the same patterns thus if a pattern appears only a finite number of times in x, it appears the same number of times in both configurations. As a consequence, Hanf lemma applies: x and y, having the same patterns, have the same theory. ∎

This theorem allows us to get an equivalence between the completeness of T_τ and a property of the tileset τ:

Theorem 4.7. *A tileset τ can produce only one tiling up to local isomorphism if and only if $T_\tau^\infty = T_\tau \cup \{\forall x, E^n(N^m(x)) \neq x | m, n \in \mathbb{Z}\}$ is complete.*

Note that the additional axioms ensure that no model of T_τ is skewed, that is all models of T_τ are based on $\mathbb{Z}^2$ or disconnected copies of $\mathbb{Z}^2$.

There is no need indeed for these additional axioms if we can ensure that the only (up to local isomorphism) tiling by τ is actually strictly aperiodic (that is has no vector of periodicity)

Proof. Before going on the proof of Theorem 4.7, we first need an extra lemma on tilings:

Lemma 4.8. *If all the tilings produced by a tileset τ are locally isomorphic then every pattern that appears in a tiling appears infinitely many times in it.*

Proof. Consider a tiling x and suppose that there exists a pattern that appears only finitely many times. By compactness, we can extract a tiling that does not contain this pattern since we can have arbitrary large patterns that do not overlap with it. The extracted tiling that does not contain this pattern will thus not have the same patterns as x. ∎

$\Rightarrow$: We prove that any two models of T_τ^∞ are elementary equivalent. This is already true for models that are tilings (Lemma 4.6) but we still have to prove it for arbitrary models. Consider two models M and M' of T_τ^∞, they are made of disconnected copies of tilings; all patterns that appear in a tiling appear infinitely many times therefore all the spheres that appear in M or M' appear infinitely many times. Thus the hypothesis of Hanf locality lemma hold, so $M \equiv M'$. Therefore T_τ^∞ is complete.

$\Leftarrow$: If T_τ^∞ is complete, for any pattern M, the formula $\exists x, \varphi_M(x)$ is either valid in any model or false in any model, therefore any two tilings contain exactly the same patterns, thus τ can produce only one tiling up to local isomorphism. ∎

Corollary 4.9. *If a tileset τ can produce only one tiling up to local isomorphism then the appeareance of any pattern is a decidable problem.*

This is a corollary of Theorem 4.7 that we express here without any model theoretic language: τ can produce only one tiling up to local isomorphism thus T_τ^∞ is complete. Given a pattern M, one can enumerate the valid proofs in T_τ^∞ and stop when either a proof of $\exists x, \varphi_M(x)$ or of $\neg\exists x, \varphi_M(x)$ is found; and such a proof will be found since T_τ^∞ is complete.

4.2.1. *On compactness.* With all those results one could try to prove some results about tilings in an elegant and short way using model theoretic arguments. Take for example the fact that any tileset that produces only periodic tilings can produce only finitely many of them [1]. This can be reformulated as "if a tileset can produce tilings with arbitrarily large periods then it can produce one that is not periodic". It is easy to write a formula ϕ_n that expresses that there is a tiling with no period lower than n. If a tileset can produce tilings with arbitrarily large periods then it has a model verifying any finite set of such formulae, thus by compactness it has a model that verifies all these formulae, e.g., it has a model that has no period. However, we can not conclude that the tileset can produce a tiling with no period. Indeed this model we obtain by compactness will certainly consist of a copy of each periodic tiling : As we have tilings of arbitrary large period, there is no common period for all these tilings, so that our model indeed does not have a period.

We would like to be able to use the compactness theorem of the first order logic but within the domain of connected models. However as said earlier, many classical theorems of first order logic will not hold. See [7] for some possible solutions.

4.3. Applying the results to model theory

A finitely axiomatizable, complete and superstable theory has been exhibited with these methods of translating tilesets into first order theories. This has historically been done by

Makowsky [9] to prove that these three properties of theories are not incompatible and then explained in a more detailed way by Poizat [11].

The idea is quite simple: take τ an aperiodic tileset that produces only one tiling up to local isomorphism; for example the one used by Berger to prove the undecidability of the domino problem [2]. Transform it in a first order theory as explained in Section 3 to obtain a finitely axiomatized theory T_τ. Since Berger proved that his tileset can not produce any tiling with a vector of periodicity, Theorem 4.7 holds without any need to add more axioms to ensure that the models are infinite by Lemma 4.4; therefore T_τ is complete and finitely axiomatizable.

We can then prove that the theory is superstable. This definition has to do with how many types there are in the theory, or more simply, with how many tilings we can produce.

It has been proven that this tileset can produce $2^{\aleph_0}$ different tilings [4, 1], therefore $2^{\aleph_0}$ countable models; Those models are not isomorphic because there is only a countable number of shifts. Furthermore, there is no skewed models, so that all models of this theory are then easy to give : they consists of some copies of these $2^{\aleph_0}$ different tilings, that is we have to say for each tiling how many times it appears. This shows that the theory is not ω-stable, but superstable.

5. Conclusion

We have seen along this paper the tight links between tilings and logic, especially between tilings properties and model theoretical properties of their interpretation. Tilings have then provided interesting examples of theories [11] as well as a good framework in which to study properties of classes of formulaes[5]

Some links still remain unexplored and might lead to interesting results. As an exemple, the Cantor-Bendixson rank [8] introduced in [1] has been motivated by the study of a notion of rank for finitely generated structures of universal theories in [7].

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