Ideal Hypothesis testing and algorithmic information transfer

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29/6/2009 Randomness Computability and Logic

Outline

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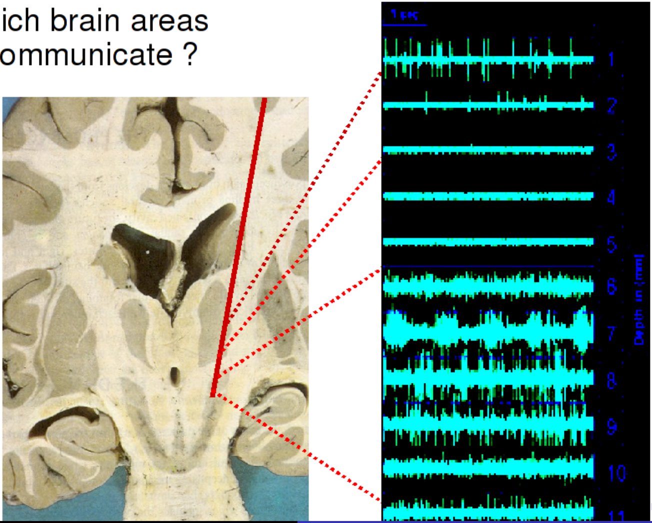
Independence and influence in timeseries

Which brain areas communicate ?



Motivation: Algorithms for independence and influence

Improvements

- in algorithms for specific *contexts*
- in general purpose algorithms

Question : Series of improved general purpose algorithms, what are they converging to ? Is there a (non-computable) limit ?

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Goal of hypothesis tests in science

Discuss inference of *probabilistic hypothesis*.

- Science = Logic with *rules* mapped from/to *observables* $\longrightarrow$ scientists *discuss* applicability in ...
- *Context* = Set of *observables* with some *observables* fixed
- *Hypothesis* = set of *rules* under discussion
- *Inference & applicability*: experiment -> larger *contexts*
- *Probabilistic rule* $\rightarrow$ *observable* not consistent in *context*.

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Procedure

- Hypotheses under discussion: $H^0 \leftrightarrow H^A$
- *Hypothesis test d*: function & critical regions
  - If $d(x)$ in critical region: **Reject** H^0
    "Either a rare event has occurred or H^0 does not describe the data"
  - otherwise: **Fails to reject** H^0
- *Simple hypothesis* = Probabilistic hypothesis fixing probabilities for all observable-values of a some observables in a context
- *Significance d for simple* H^0 = Probability H^0 is **rejected** according to H^0
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Optimal ratio-test

- One-sided test d, critical region $> \lambda$
- Significance en Sensitivity:

$$\alpha(\lambda) = \sum\{P^0(y) : d(y) \geqslant \lambda\}$$
$$\beta(\lambda) = \sum\{P^A(y) : d(y) \leqslant \lambda\}$$

- Neyman-Pearson': $\beta \circ \alpha^{-1}$ is uniformly maximal iff d is a.e. monotone function of the ratio-test:

$$P^A(x)/P^0(x).$$

- A-priori belief in H^0, H^A is a^0, a^A, Bayesian a-posteriori belief:

$$\frac{a^0}{a^A}\frac{P^0(x)}{P^A(x)}.$$

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Composite hypothesis tests

- *Composite hypotheses* = set of simple hypotheses
- Different approaches theoretically or empirically optimal in different context for H^0, H^A composite:
  1. *Uniformly optimal test*: optimal $\beta \circ \alpha^{-1}$ → few cases
  2. *Bayesian approach* → subjective
     uniform: Bonferroni-correction
  3. *Generalized maximum likelihood*

$$\frac{\max\{P(x) : P \in H^A\}}{\max\{P(x) : P \in H^0\}}.$$

  4. *Information theory*

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Outline

Length conditional semimeasures

P is length conditional iff

$$\forall n : \sum \{P(x) : x \in 2^n\} \leqslant 1$$

- No nice coding result for monotones semimeasures (Gacs, previous talk)
- For most experiments length of the measurements contains very little information

"length conditional semimeasure" abbr. to "semimeasure"

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Universal semimeasures

Definition

Semimeasures P, Q,

- *P dominates Q* ($P \geqslant^* Q$) iff $\exists c \forall x : P(x) \geqslant cQ(x)$
- $P =^* Q$ iff $P \leqslant^* Q$ and $P \geqslant^* Q$
- *m is universal in H* iff $\forall P \in H$: $P \leqslant^* m$
- *Ratio test of H^0, H^A*:

$$d(x) = m^A(x)/m^0(x),$$

- Generalized maximum likelihood testing (up to *-constant)
- Enumerable cases: Bayesian approach $m^A =^* \sum a_i P_i$

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Interpretation

For H^0 enumerable, for $i < k/(logk)^2$:

$$\frac{P_i^0(x)}{m^A(x)} \leqslant^* \frac{km^0(x)}{m^A(x)}$$

A high $d(x)$ means that either:

- A complex model from the zero hypothesis describes data x
- The alternate hypothesis m^A better describes data x
- A rare event has occurred

$\rightarrow$ Theoretical 'significance' is lower than $1/d(x)$
$\rightarrow$ In practice, (ex. causality, independence), the first interpretation cannot be completely eliminated, therefore: significance must be defined empirically in context.

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1. Composite Hypothesis tests
   - Motivation
   - Procedure
   - Ratio-test of universal semimeasures
   - Criterion for universality
2. Influence tests
   - Causal semimeasures
   - Total conditional coding theorem
   - Incremental coding theorem
3. Decompositions
   - Decompositions of algorithmic complexity
   - Decompositions of mutual information
4. Randomness tests in hypotheses
   - Sumtests
   - Independence tests

Definitions

Let S be a set of enumerable semimeasures

- S is *testable* iff for some computable logic expression L:

$$P \in S \Leftrightarrow \forall t, n \leqslant t : L(P_t^n).$$

with P_t^n = finite restriction of P_t on 2^n

- S is *convex* iff $\forall P, Q \in S, a \in [0,1] : aP + (1-a)Q \in R$
- S has a *computable monotone convex upper-bounded* iff a computable R exists, such that for all a, P, Q, P', Q'

$$aP + (1-a)Q \leqslant R(P, Q, a)$$

$$P' \geqslant P; Q' \geqslant Q \Rightarrow R(P', Q', a) \geqslant R(P, Q, a)$$

Remark: S is *convex*, implies S is *computable monotone convex upper-bounded*.

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with P_t^n = finite restriction of P_t on 2^n

- S is *convex* iff $\forall P, Q \in S, a \in [0,1] : aP + (1-a)Q \in R$
- S has a *computable monotone convex upper-bounded* iff a computable R exists, such that for all a, P, Q, P', Q'

$$aP + (1-a)Q \leqslant R(P, Q, a)$$

$$P' \geqslant P; Q' \geqslant Q \Rightarrow R(P', Q', a) \geqslant R(P, Q, a)$$

Remark: S is *convex*, implies S is *computable monotone convex upper-bounded*.

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$\Sigma =$ *set of enumerable semimeasures*

(i) *If S is testable,* $P^{(0)} = 0 \in S$*, than* $S \cap \Sigma$ *is enumerable.*

(ii) *If S has a computable monotone convex upper-bounded, and* $S \cap \Sigma$ *is enumerable, than it has a universal element.*

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Explicit construction of universal element of S

- Enumeration: $S = P_0, P_1, ...$
- Mixture

$$m^S(x) = \sum a_i P_i$$

$\rightarrow$ Very difficult to approximate

Hypotheses with universal enumerable semimeasure:

- Semimeasures
- Conditional semimeasures
- Uniform conditional semimeasures (further)
- Independent semimeasures: $P(x, y) = Q(x)R(y)$
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Outline

Notation and Definitions(1)

From now on: $x, y, z \in 2^n$
$x^i = x_1 x_2 \ldots x_i$
If $v \in 2^i$ then $P(v|y) = \sum\{P(vw) : w \in 2^{n-i}\}$
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- A conditional semimeasure P is *conditional causal* ($P(x\lceil y\uparrow)$) iff

$$\forall i, x, y, z : y^i = z^i \Rightarrow P(x^{i+1}|y) = P(x^{i+1}|z)$$

- The *associated causal semimeasure $P(x|y\uparrow)$ associated with a semimeasure $P(x,y)$* is:

$$P(x|y\uparrow) = \frac{P(x^1, y^0)}{P(x^0, y^0)}\frac{P(x^2, y^1)}{P(x^1, y^1)}\cdots\frac{P(x^n, y^{n-1})}{P(x^{n-1}, y^{n-1})}$$

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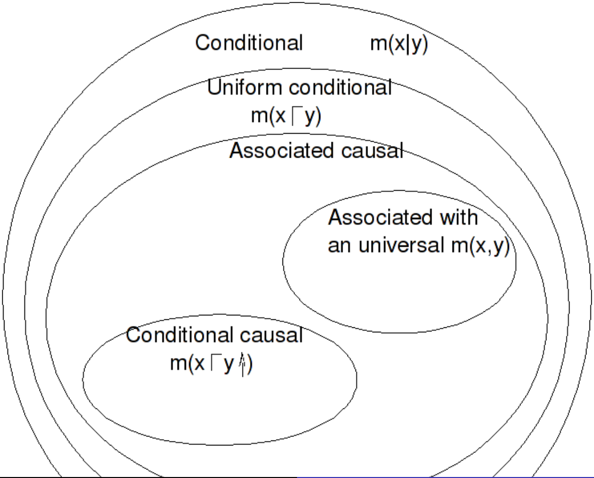
Relations between conditional and causal semimeasures



Unstable inference of influence

Proposition

For enumerable P, with $P(x, y) \geqslant^* 2^{-2n}$, there are enumerable Q, R with $P =^* Q =^* R$ and

$$\log \frac{Q(x|y\uparrow)}{P(x|y\uparrow)} > o(n) \qquad \log \frac{P(x|y\uparrow)}{R(x|y\uparrow)} > o(n).$$

Corollary

The set of associated causal semimeasures associated with $P(x, y) \geqslant 2^{-2n}$, has no universal element.

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- The ratio-test:

$$\frac{m(x|y)}{m(x|y\uparrow)}$$

depends on the choice of m.
- Connection to observed instability of influence measure from Huffman trees ?
- Inference of influence with ideal compression using K (coding theorem) either:
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Outline

Theorem (Coding)

$$K(x) =^+ \log m(x)$$
$$K(x|y) =^+ \log m(x|y)$$

Apply search-heuristics for data compression to estimate $m(x), m(x|y)$

(Length conditional prefix-free) Kolmogorov complexity

$\langle .,. \rangle \rightarrow$ computable bijective pairing function.

$$
\begin{aligned}
K_t(x) &= \min\{l(p) : \Phi_t(p,n)\downarrow = x\} \\
K(x) &= lim_{t\to\infty} K_t(x) \\
K(x,y) &= K(\langle x,y\rangle) \\
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Difference total conditional and conditional complexity

Proposition

For all n there are $x, y \in 2^n$ with:

$$K(x\lceil y) - K(x|y) \geqslant^+ n$$

Difference due to Halting information

Proposition

$$K(x\lceil y) - K(x|y) \leqslant^+ K(y) - K'(y) + O(\log k_{xy})$$

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m-depth

Definition

$$t_k = \min\{t : m(\epsilon) - m_t(\epsilon) \leqslant 2^{-k}\}$$

$$k_{xy} = \min\{k : K_{t_k}(x, y|n^*) =^+ K(x, y|n^*)\}$$

Lemma

Let $m^i, i = 0, 1$ be two universal semimeasures
Let k^i_{xy} be the corresponding m^i-depths, then:

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Proposition (Total coding theorem)

$$K(x\lceil y) = \log m(x\lceil y) \pm O(\log k_{xy})$$

Outline

Incremental computation

$x^i = x_1 \ldots x_i$
$\Phi(p, x \uparrow) \downarrow= y$ iff $\forall i < n : \Phi(p, x^i, n) =\downarrow y_{i+1}$

x	y
=	=
x_4	y_4
x_3	y_3
x_2	y_2
x_1	y_1

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Incremental complexity

Definition

- *total conditional complexity (remind)*
  $K(x\lceil y) = \min\{l(p) : \Phi(p,y)\downarrow = x \wedge \forall z \in 2^n : \Phi(p,z)\downarrow\}$
- *incremental conditional complexity*
  $K(x|y\uparrow) = \min\{l(p) : \Phi(p,y\uparrow)\downarrow = x\}$.
- *total incremental conditional complexity*
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Proposition (Incremental coding bound)

If P is a computable causal semimeasure, then

$$-\log P(x|y\uparrow) \geqslant^+ K(x\lceil y\uparrow).$$

Proposition (Incremental coding theorem)

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Outline

Well known decomposition

Theorem

$$K(x,y) =^+ K(x) + K(y|x^*)$$

with $x^* =$ *shortest description for* x*, witness of* $K(x)$

Incremental$^+$ complexity: divergence

$$
\begin{aligned}
K(x|y\uparrow^+) &= \min\{l(p) : \Phi(p, y\uparrow)\downarrow = x\} \\
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Trivial: x, y has constant bounded m-depth:

$$K(x|y\uparrow) + K(y|x\uparrow^+) =^+ K(x, y)$$

Proposition

$\exists c > 0 \forall n \exists x, y \in 2^n$:

$$K(x|y\uparrow) + K(y|x\uparrow^+) - K(x, y) \geqslant^+ cn.$$

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Total conditional complexity: decomposition

Notation:

$$K(y\lceil x\uparrow^+, p) = \min\{l(q) : \Phi(q, x\uparrow^+, p)\downarrow y, \forall z \Phi(q, z\uparrow^+, p)\downarrow\}$$

Proposition

Let p be the minimal program in the definition of $K(x\lceil y)$*:*

$$K(x\lceil y) + K(y|p) \leqslant^+ K(x,y) + O(\log k_{xy}).$$

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Question

$$K(x|y\uparrow) + K(y|x\uparrow^+, p) =^+ K(x,y) + O(\log k_{xy})$$

with p the minimal program in $K(x|y\uparrow)$.

Generalization ?

Question

Let $\mathcal{S}$ be an enumerable set of either finite or infinite, computable or enumerable sets in ω. Let:

$$S_x = \arg\min\{K(S) : x \in S \in \mathcal{S}\}$$

Does the following equation hold ?

$$K(S_x) + K(x|S_x^*) =^+ K(x)$$

where $K(S) = I(S^*)$ and S^* is either a minimal program that enumerates all elements of S_x and halts, or a minimal program that enumerates all elements of S_x and possibly continues computing.

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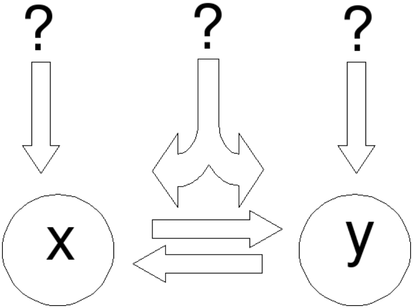
Outline

Algorithmic mutual information

$$
\begin{aligned}
I(x;y) &\equiv K(x) - K(x|y^*) \\
&=^+ K(x) + K(y) - K(x,y) \\
&=^+ I(y;x) \\
&=^+ \log \frac{m(x,y)}{m(x)m(y)}
\end{aligned}
$$

Interpretation of I as ratio-test for independence (Levin).

Time series



- 'Origin' of $K(x, y)$ from three sources.
- $I(x; y)$ as a sum of (the three arrows in the middle):
  - Information flow from x to y.
  - Information flow from y to x.
  - Information from a common source.

Information transfer and instantaneous common information

total information transfer

total conditional instantaneous common information

$IT(x \leftarrow y) = K(x) - K(x|y\uparrow)$

$TIT(x \leftarrow y) = K(x) - K(x\lceil y\uparrow)$

$IT(x\uparrow; y\uparrow) = K(x|y\uparrow) - K(x|y\uparrow^{+})$

$TIT(x\uparrow; y\uparrow) = K(x\lceil y\uparrow) - K(x\lceil y\uparrow^{+})$

$ITc(x\uparrow; y\uparrow) = K(x|y\uparrow) - K(x|y\uparrow^{+}, p)$

$TITc(x\uparrow; y\uparrow) = K(x\lceil y\uparrow) - K(x\lceil y\uparrow^{+}, p)$

with p the withness of $K(x|y\uparrow)$ $K(x\lceil y\uparrow)$

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Decomposition of mutual information

Corollary

$$TIT(y \leftarrow x) + TIT(x \leftarrow y) + TITc(x \uparrow; y \uparrow) =^+ I(x; y) \pm O(\log k_{xy})$$

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Question

How symmetric is $ITc(x \uparrow; y \uparrow)$?

For multi-symbol tapes:

$$I(x \uparrow; y \uparrow) - I(y \uparrow; x \uparrow) \geqslant o(\log n).$$

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Outline

Sumtest

Let P be a semimeasure over ω.

$d : \omega \cup \{-1\} \to \omega \cup \{-1\}$ is a P-sumtest iff

$$\sum_{x \in \omega} P(x) 2^{d(x)} \leqslant 1$$

- Identity testing: “Is x typical for P ?”
- Largest d in some computability class ?

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Universality

- A function f dominates g ($f \geqslant^+ g$) iff
$$\exists c \forall x : f(x) + c \geqslant g(x).$$
- A function f is universal in a set S iff $f \in S$ and
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Universality theorem

Proposition

Let d be an m-sumtest,

- *if d is computable or enumerable than*

$$d(x) \leqslant^+ 2K(d),$$

- *if d is co-enumerable than*

$$d(x) \leqslant^+ \log l(x) + 4 \log \log l(x),$$

  and

$$\exists d' \forall n \exists x, y \in 1^n : d'(x,y) - d(x,y) \geqslant^+ \log l(x) - O(\log^1 l(x)).$$

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Subgoals of proof

Given: co-enumerable m-sumtest d
Required: x and d' such that $d(x) - d'(x) \geqslant \ldots$

- Few x have high $d_0(x)$ (sumtest).
- Choose x such that $m_t(x)$ remains constant and increases at large $t = s$: $K^+(s) >> K(d)$.
- d_0 can impossibly distinguish x in a set of y with constant $m_t(y)$. Therefore, $d_0(x)$ is low.
- Construct d' that knows s and therefore can reserve a high $d'_0(x)$
- ... iterate ...

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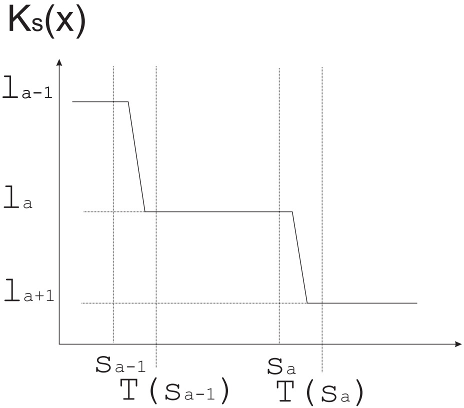
Gradual compressible strings



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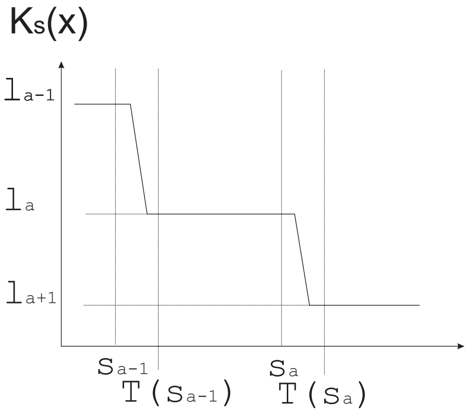

Constructed x has high minimal sufficient statistics.
Any co-enumerable test bounded by minimal sufficient statistic within logarithmic term.

Outline

Independence test

Definition

- $d : \omega \cup \{-1\}^2 \to \omega \cup \{-1\}$ is a (P, Q)-independence test iff

$$\sum_{x,y \in \omega} P(x)Q(y)2^{d(x,y)} \leqslant 1.$$

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Universality theorem

Proposition

Let d be an (m,m)-independence test,

- *if d is computable or enumerable, then*

$$d(x,y) \leqslant^+ 2K(d)$$

- *if d is co-enumerable,*

$$\exists d' \forall n \exists x, y \in 2^n : d'(x,y) - d(x,y) \geqslant^+ \log l(x) - O(\log^2 l(x)),$$

$$d'(x,y) - d(x,y) \geqslant^+ l(x) - O(\log l(x)).$$

Summary

- Influence: Ideal limit point for improving sequence of algorithms exist
  Independence: Ideal limit point does not exist
- Decomposition for Kolmogorov complexity and mutual information
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