# Randomness and genericity

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# Introduction

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- Every Schnorr random real is Kurtz random.
- Every weakly 1-generic real is Kurtz random.

How closely related can members of these classes be?

# Definitions

A Turing machine is *computable* if the measure of its domain is a recursive real.

The prefix-free Kolmogorov complexity of a string $\sigma$ relative to a particular prefix-free Turing machine $M$ is

$$K_M(\sigma) = \min\{|\tau| \mid M(\tau) = \sigma\}.$$

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A real $A$ is *Schnorr random* if for every computable Turing machine $M$,

$$(\exists c)(\forall n)[K_M(A\restriction n) \geq n - c].$$

# Definitions

A real $A$ is *n-generic* if for every $\Sigma^0_n$ sentence $\varphi$, some initial segment of $A$ either forces $\varphi$ to be true or has no extensions for which it is true.

A real $A$ is *weakly* 1-*generic* if for every dense r.e. open set $D \in 2^\omega$, $A \in D$.

# Previous results

**Theorem**

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**Theorem (Demuth and Kučera)**

*No 1-generic real Turing computes a Martin-Löf random real.*

**Theorem (Nies, Stephan, and Terwijn)**

*Every 2-generic real forms a minimal pair with every 2-random real.*

# Turing degrees and 1-generics

**Theorem**

*There is a 1-generic real that is Turing equivalent to a Schnorr random real.*

**Proof.**

Simply consider a high 1-generic. □

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### Theorem

*No nonhigh 1-generic real Turing computes a Schnorr random real.*

### Proof.

Any nonhigh Schnorr random real is Martin-Löf random and thus fixed-point free, so everything above it must be FPF as well. However, no 1-generic is FPF. □

# Turing degrees and 2-generics

**Theorem**

*If $G$ is a 2-generic real and $A \leq_T G$, $A$ cannot be Schnorr random.*

We will prove that $A$ is not Schnorr random by showing that we cannot force it to be Schnorr random.

# Proof

Let $G$ be a 2-generic, and suppose that $A \leq_T G$ via $\Psi$.

The statement "$\Psi$ is total" is a $\Pi^0_2$ statement, and $G$ must force this to hold. Let $p \subseteq G$ be the initial segment forcing this.

Our forcing conditions will be $\mathcal{P} = \{p_i \mid p \subseteq p_i\}$. If we apply $\Psi$ to each member of this set, we get

$$T = \{r_i \mid r_i = \Psi(p_i)\}.$$

# Proof

Recall: $A$ is not Schnorr random if there is some computable machine $M$ such that

$$(\forall c)(\exists n)[K_M(A\restriction n) < n - c].$$

We will build an $M$ such that for each $r_i$ and $c$, the statement

$$(\exists r_j \supseteq r_i)(\exists n)[K_M(r_j\restriction n) < n - c]$$

will hold.

# Proof

Stage 0:

- Set $M = \emptyset$.

Stage $\langle i, c \rangle$:

- Find some $n$ larger than every $n$ previously considered such that $\langle i, c \rangle < n - c$.
- Find some $r_j \supseteq r_i$ such that $|r_j| \geq n$.
- Set $M(\langle i, c \rangle) = r_j \upharpoonright n$.

# Proof

We first note that $\mu(\text{dom}(M)) = \sum_{n \geq 1} \frac{1}{2^n} = 1$, so $M$ is a computable Turing machine.

Above each $r_i$, for all $c$, we can find some extension $q$ of length $n$ such that $K_M(q) = \langle i, c \rangle < n - c$. Therefore, $G$ cannot force Schnorr randomness at any point, so $A$ must not be Schnorr random.

# The *tt*-degrees and 1-genericity

**Theorem**

*If $G$ is a 1-generic real and $A \leq_{tt} G$, then $A$ is not Schnorr random.*

**Proof.**

By the previous argument. □

# The *tt*-degrees and hyperimmunity

Recall: A set $B$ is hyperimmune if it is infinite and there is no recursive function that dominates $p_B$.

**Theorem**

*If $B$ is hyperimmune and $A \leq_{tt} B$ via $\Psi$, then $A$ is not Schnorr random.*

# Definitions

A martingale is a a function $d : 2^{<\omega} \to \mathbb{R}^+ \cup \{0\}$ such that for all $\sigma \in 2^{<\omega}$,

$$d(\sigma) = \frac{d(\sigma 0) + d(\sigma 1)}{2}.$$

A martingale is *recursive* if the reals $d(\sigma)$ are uniformly recursive.

We say that a martingale $d$ *succeeds* on a real $A$ if

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A real $A$ is Schnorr random if there is no recursive martingale $d$ such that $d(A{\upharpoonright}n) \geq f(n)$ infinitely often for some unbounded, nondecreasing recursive function $f$.

# Proof of theorem

To prove this, we build a recursive martingale that succeeds on $A$.

- Partition $\omega$ into intervals $\langle I_n \rangle$ such that the length of $I_n$ is $2n + 1$.
- For each $n$, define a martingale $d_n$ as follows:
  - the initial capital of $d_n$ is $\frac{1}{2^{n+1}}$ and
  - $d_n$ bets on $\Psi(D_n)$ on the interval $I_n$.
- Let $d$ be the martingale that behaves like $d_n$ on each interval $I_n$.

## Fact

*Suppose $B$ is hyperimmune and $f$ is a recursive function. Then there are infinitely many $n$ such that*

$$B \cap \{0, \ldots, f(n)\} = D_n \cap \{0, \ldots, f(n)\},$$

*where $D_n$ is the $n^{th}$ canonical finite set.*

Define $f$ so $f(n)$ is the use of $\Psi$ for all elements in $I_n$.

- If $B \cap \{0, \ldots, f(n)\} = D_n \cap \{0, \ldots, f(n)\}$, $d$ earns $\frac{1}{2^{n+1}} \cdot 2^{2n+1} = 2^n$ on $I_n$.
- If $B \cap \{0, \ldots, f(n)\} \neq D_n \cap \{0, \ldots, f(n)\}$, $d$ loses at most $\frac{1}{2^{n+1}}$ on $I_n$.

Furthermore, we can see that $d$ succeeds on $B$ "quickly enough."

# The *wtt*-degrees and hyperimmunity

**Theorem**
*If $A$ is not high, $B$ is hyperimmune, and $A \leq_{wtt} B$, then $A$ is not Schnorr random.*

**Theorem**
*If $B$ is a high 1-generic real, then there is a Schnorr random real $A$ such that $A \leq_{wtt} B$.*

# Proof

If $B$ is 1-generic and high, we can enumerate a list of indices $\langle e_i \rangle$ so that

- each $m_{e_i}$ is a total rational-valued martingale and
- each total martingale is indexed by some $e_i$.

We can do this in such a way that $m_{e_i}$ is determined by only the first $i$ bits of $B$.

Replace each $m_{e_i}$ by $d_i$, where $d_i(\sigma) = 1$ for $|\sigma| < i$ and

$$d_i(\sigma^\frown\tau) = \frac{1 + m_{e_i}(\sigma^\frown\tau)}{1 + m_{e_i}(\sigma)}$$

when $|\sigma| = i$.

# Proof

Let $d(\sigma) = \sum_k \frac{1}{2^{k+1}} d_k(\sigma)$. This is a rational-valued martingale that is *wtt*-computable from $A$.

Now we define our $A \leq_{wtt} B$ inductively: let $A(k+1)$ be 0 if

$$d(A{\upharpoonright}k^\frown 0) \leq d(A{\upharpoonright}k)$$

and 1 otherwise.

Clearly, $d$ will not succeed on $A$, so $A$ is recursively random and thus Schnorr random.

Merci!