Effective Wadge Hard sets

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Classical Wadge theory, 1972

Wadge reducibility

X, Y topological spaces $\quad A \subseteq X,\ B \subseteq Y$

$$A \ \leq_W B \iff A = f^{-1}(B)$$

for some continuous map $f : X \to Y$

Topological version of many-one and polynomial time reducibility between sets of integers, words,. . . .

Classical Wadge theory

In case the space is

zero-dimensional (= basis of clopen sets)

Polish (= complete, metric, countable dense subset)

like Cantor space $\mathbf{2}^\omega$, Baire space $\mathbb{N}^\omega$

(product of discrete topologies)

Wonderful properties

- Borel hierarchy: $\boldsymbol{\Sigma}^0_\alpha \setminus \boldsymbol{\Pi}^0_\alpha$ is a Wadge degree
- Hausdorff difference hierarchy inside $\boldsymbol{\Delta}^0_\alpha$: idem
- Wadge degrees: well-founded almost linear order

Wadge theory in Scott $\mathcal{P}(\mathbb{N})$

In this talk we shall deal with

$$\mathcal{P}(\mathbb{N}) \qquad \text{with Scott topology}$$

Basis of open sets: $B_A = \{X \mid X \supseteq A\}$ for A finite
Positive information topology

Non Hausdorff T_0 space
($=$ given two distinct points in the space
there is an open set containing only one of them)

Hausdorff hierarchy

Finite levels: finite unions of differences of open sets

$=$ in an increasing chain take elements with even (resp. odd) proper rank

Extends to the transfinite

Wadge theory in Scott $\mathcal{P}(\mathbb{N})$

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($=$ given two points there is an open set
containing only one of them)

in $\mathcal{P}(\mathbb{N})$ wonderful Wadge theory fails

SO WHY LOOK AT IT ANYWAY?

Our motivation

Becher & Chaitin, 2002
computations on infinite inputs $\mathbf{2}^\omega \to \mathbf{2}^{\leq\omega}$

Proba(*finitely many symbols are read*
$+$ *finite output*) 2-**random**

Proba(*finitely many symbols are read*
$+$ *output codes a cofinite set*) 3-**random**

Our motivation

Becher & Grigorieff, 2005-2009

comput. on infinite inputs $\begin{cases} \mathbf{2}^\omega \to \mathbf{2}^{\leq\omega} \\ \mathbf{2}^\omega \to \mathcal{P}(\mathbb{N}) \end{cases}$

Mathematically significant 2, 3, 4,... random reals, e.g.,

Proba(*output a pair* (X, Y) *s.t.* $X \leq_{Turing} Y$)

4-**random**

Our motivation

Theorem behind our randomness results

$U : \mathbf{2}^\omega \to \mathcal{P}(\mathbb{N})$ machine universal by prefix adjunction

$+\ \mathcal{O}$ is $\Sigma^0_n(\mathcal{P}(\mathbb{N}))$

$+\ \mathcal{O}$ **effectively Wadge hard for** $\Sigma^0_n(\mathbf{2}^\omega)$

(for continuous maps Scott $\mathcal{P}(\mathbb{N})$ *to* $\mathbf{2}^\omega$*)*

$\Rightarrow \qquad Proba(U^{-1}(\mathcal{O}))$ n-**random**

Also extension to almost Wadge hardness

Effectivize Wadge theory

Wadge theory with Scott $\mathcal{P}(\mathbb{N})$

Effective Wadge Theory? There are problems

Continuous reductions in Wadge theory

$=$

winning strategies in Wadge games

For reductions between Borel sets:

use Borel determinacy

Highly non effective reductions:

outside the effective Borel hierarchy

Effective Wadge Theory? There are problems

Replace continous maps by computable maps

Turing machine on infinite inputs $F : \mathbf{2}^\omega \to \mathbf{2}^\omega$

Problem: F total is Π^0_2 condition on machine code
(use König's Lemma)

- **either accept partial functions $\mathbf{2}^\omega \to \mathbf{2}^\omega$**

- **or change the target space** $F : \mathbf{2}^\omega \to \mathbf{2}^{\leq\omega}$
put Scott topology on $\mathbf{2}^{\leq\omega}$ (Non Hausdorff T_0)

Open basis: $B_s = \{\xi \in \mathbf{2}^{\leq\omega} \mid s \leq_{\text{prefix}} \xi\}$
for $s \in \mathbf{2}^{<\omega}$

Effective Wadge Theory? There are problems

$F : \mathbf{2}^\omega \to \mathcal{P}(\mathbb{N})$

$F(\alpha) = \{n \mid 0\overbrace{1\dots1}^{n \text{ times}}0 \text{ is a factor of } \alpha\}$

F is computable

F continuous? OK with Scott topology on $\mathcal{P}(\mathbb{N})$

In general, Scott topology (1972)
on directed complete partial orders (dcpo's)

(Effective) Wadge theory on $\mathcal{P}(\mathbb{N})$ with Scott topology

Extension of classical Wadge theory to $\mathcal{P}(\mathbb{N})$

A.Tang, 1978, 1981 ($+$ Wadge)
V. Selivanov, 2002-2006

First steps of effectivization

V. Selivanov, 2002-2006

Main result

Theorem (Hardness and chains)
(Effective) Wadge hardness characterized in terms of alternating chains.

Alternating chains

In Scott domains we can compare the amount of positive information.
In $\mathcal{P}(\mathbb{N})$ this is set inclusion.

With positive and negative information,
this makes no sense (e.g., in Cantor or Baire)

Definition. Let $\mathcal{H} \subseteq \mathcal{P}(\mathbb{N})$, α an ordinal.
$(X_\beta)_{\beta \leq_\alpha}$ increasing (decreasing) $\mathcal{H}$-alternating chain if

$$X_\beta \in \mathcal{H} \iff \beta, \alpha \text{ have different parities}$$

In particular, $X_\alpha \notin \mathcal{H}$.

2-chains

Proposition. (Hardness for level 1)
$\mathcal{H} \subseteq \mathcal{P}(\mathbb{N})$ Wadge hard for $\boldsymbol{\Sigma}^0_1(\mathcal{P}(\mathbb{N}))$
iff $\exists$ decreasing $\mathcal{H}$-alternating 2-chain

In case $\mathcal{H}$ is open, the chain can be taken $(\text{finite}, \emptyset)$

Effective Wadge hardness for $\Sigma^0_1(\mathcal{P}(\mathbb{N}))$:
require two r.e. sets

Finite chains

Proposition.(Hardness for finite levels Hausdorff hierarchy).
Finite decreasing alternating chains of sets alternating in $\mathcal{H} \subseteq \mathcal{P}(\mathbb{N})$

Same holds for effective Wadge hardness in the lightface case requiring that the sets be r.e.

Decreasing $(\alpha+1)$-chains

Theorem.(Hardness for D_α).
$\mathcal{H} \subseteq \mathcal{P}(\mathbb{N})$ α be an ordinal

$\mathcal{H}$ is Wadge hard for $\underset{\sim}{\mathbf{D}}_\alpha$ (difference hierarchy)
iff
$\exists$ decreasing $\mathcal{H}$-alternate $(\alpha+1)$-chain.

Effective Wadge hardness:
Adding computability to the characterization above gives effective Wadge hardness for the class D_α.

Alternating distributive lattice

$(\mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N}), \subseteq)$ boolean algebra

$\mathcal{X} = \{(U, V) \mid V \subseteq U\}$
bounded distributive sublattice (not complemented)

Definition.(Alternating distributive lattice).
$(X_i)_{i \in \mathbb{N}}$ sequence of disjoint finite sets

$$\ell : \mathcal{X} \to \mathcal{P}(\mathbb{N}) \qquad \ell(U, V) = \bigcup_{n \in U} X_{2n} \cup \bigcup_{m \in V} X_{2m+1}$$

ℓ **alternating in** $\mathcal{H} \subseteq \mathcal{P}(\mathbb{N})$
iff $\quad \ell(U, V) \in \mathcal{H} \Leftrightarrow U \neq V$

Alternating distributive lattice

Theorem. $\mathcal{H}$ is $\utilde{\mathbf{\Sigma}}^0_2(\mathcal{P}(\mathbb{N}))$ Wadge hard
iff
$\exists (X_i)_{i \in \mathbb{N}}$ disjoint finite sets such that
ℓ is alternating in $\mathcal{H}$

Effective Wadge Σ^0_2 hard : require an r.e. sequence

Fact.
$\mathcal{S}_2 = \{Z \mid \exists n\ (2n \in Z \wedge 2n+1 \notin Z\}$
is Wadge $\utilde{\mathbf{\Sigma}}^0_2$-complete
effectively Wadge Σ^0_2-complete.

Classical complete sets in Cantor fail in $\mathcal{P}(\mathbb{N})$

$\mathit{FIN} \subseteq \mathbf{2}^\omega$ is $\Sigma^0_2(\mathbf{2}^\omega)$ and hard for $\boldsymbol{\Sigma}^0_2(\mathbf{2}^\omega)$.

$\mathcal{P}_{fin}(\mathbb{N})$ is $\Sigma^0_2(\mathcal{P}(\mathbb{N}))$ and hard for $\boldsymbol{\Pi}^0_1(\mathcal{P}(\mathbb{N}))$
but not for $\Sigma^0_1(\mathcal{P}(\mathbb{N}))$.

No subfamily of $\mathcal{P}_{fin}(\mathbb{N})$ is $\Sigma^0_2(\mathcal{P}(\mathbb{N}))$ complete
(nor for differences of $\Sigma^0_1(\mathcal{P}(\mathbb{N}))$)

$R.E.$ is not hard for $\Sigma^0_1(\mathcal{P}(\mathbb{N}))$.

Transfer hard for $\mathcal{C}(\mathcal{P}(\mathbb{N}))$ to hard for $\mathcal{C}(\mathbf{2}^\omega)$

Proposition.
$\mathcal{C}$ Borel class

If $\mathcal{H} \subseteq \mathcal{P}(\mathbb{N})$ Wadge hard for $\mathcal{C} \subseteq \mathcal{P}(\mathbb{N})$
then $\mathcal{H}$ is Wadge hard for $\mathcal{C}(\mathbf{2}^\omega)$.

Effective Wadge hardness: similar.

Another negative result

Proposition. No set in $\mathbf{2}^{\leq\omega}$ Wadge hard for $\utilde{\mathbf{\Sigma}}^0_2(\mathcal{P}(\mathbb{N}))$.

Proof. It is required an $\omega + 2$ chain in $\mathbf{2}^{\leq\omega}$ to mimic the alternating condition. But the longest chain in $\mathbf{2}^{\leq\omega}$ has length $\omega + 1$.

Thank you for your attention