Feeble subsets

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Randomness with respect to a measure

Martin-Löf randomness for arbitrary measures on 2^ω:

Definition 1

A *test for μ-randomness* is a uniformly $\Sigma^0_1(\mu)$ sequence $\{U_n\}_{n\in\omega}$ with $\mu(U_n) \leq 2^{-n}$.

The test $\{U_n\}_{n\in\mathbb{N}}$ defines a null set $\cap_n U_n$. X passes the test for randomness $\{U_n\}_{n\in\mathbb{N}}$ if $X \notin \cap_n U_n$. If X passes all μ-randomness tests then X is *μ-random*.

Bernoulli measures

For each $n \in \omega$,

$\mu_p(\{X : X(n) = 1\}) = p$

$\mu_p(\{X : X(n) = 0\}) = 1{-}p$

and
$X(0), X(1), X(2), \ldots$
are mutually
independent random
variables.

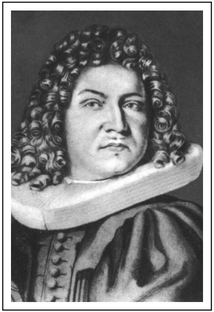

Figure: Jakob Bernoulli

Example

The Strong Law of Large Numbers for μ_p states that for almost all X according to the measure μ_p, we have

$$\forall\varepsilon > 0\,\exists N\forall n > N\,\left|\frac{\textit{the \# of 1s up to n in X}}{n} - p\right| < \varepsilon.$$

Suppose X does not satisfy the SLLN, as witnessed by a number ε_0. Let

$$U_N = \{Z : \exists n > N\,\left|\frac{\textit{the \# of 1s up to n in X}}{n} - p\right| \geq \varepsilon_0\}.$$

Then U_N is open and in fact $\Sigma^0_1(\mu_p)$; and $\mu(U_N)$ goes effectively to 0. Thus, X is not Martin-Löf random with respect to μ_p.

Definition 2

A set $Y \subseteq \mathbb{N}$ is *feeble* if it is infinite, and no Martin-Löf random set is computable from Y.

Theorem 3 (Law of Feeble Subsets)

Almost every $X \subseteq \mathbb{N}$, according to $\mu := \mu_{1/2}$, has a feeble subset $Y \subseteq X$.

(Passing from X to Y we suffer a “loss of randomness beyond algorithmic repair.”)

Theorem 4 (The Law of Feeble Subsets is Arithmetical)

Every X that is ***Martin-Löf random relative to*** $0'$ *has a feeble subset* $Y \subseteq X$.

Example

Let X be Martin-Löf random and let Y be a computably chosen subset of X. Say,

$$Y = \langle X(0), 0, X(2), 0, X(4), 0, \ldots \rangle.$$

Then Y is an infinite subset of X, but Y does compute a Martin-Löf random set, namely

$$Z = \langle X(2), X(4), X(6), X(8), \ldots \rangle$$

so Y is not feeble.

Generally, if X is Martin-Löf random and R is computable ($R \leq_T \emptyset$) then $X \cap R$ is not feeble.

Question 1

If there a Martin-Löf random set X and a set $R \leq_T X$ such that $R \cap X$ is feeble?

Partial negative result: such an X can not be hyperimmune-free.

von Neumann's randomness extractor

Let X be Martin-Löf random let Y be a "randomly chosen" subset of X. That, is each 1 in X is converted to a 0 with probability $\frac{1}{2}$. Then Y *does* compute a Martin-Löf random set. Namely, let Z be obtained from X by making the following replacements:

$$\langle X(2n), X(2n+1)\rangle \mapsto Z(n)$$

$$\langle 0,0\rangle \mapsto \langle\rangle$$

$$\langle 1,1\rangle \mapsto \langle\rangle$$

$$\langle 0,1\rangle \mapsto \langle 0\rangle$$

$$\langle 1,0\rangle \mapsto \langle 1\rangle$$

How to not find feeble subsets

Let X be Martin-Löf random.

- A feeble subset cannot be obtained by intersecting X with a computable set.
- A feeble subset cannot be obtained by taking a μ_p-random subset of X.
- A feeble subset cannot be obtained by taking a sufficiently sparse subset of X.

How to find feeble subsets

- (K., 2007) Show that

  $$\text{BN1R} := \{Y : Y \text{ computes no Martin-Löf random set}\}$$

  is a large subclass of 2^ω in terms of Hausdorff dimension, and that therefore the subsets of X must intersect it.
- (Joe Miller, 2008): Each DNC function computes an infinite subset of a Martin-Löf random set. Combine this with Kumabe 1993: not every DNC function computes a Martin-Löf random set.
- (K., 2008) Let Y be a randomly selected member of a large[1] closed subclass of BN1R, the random choice being carried out by X.

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[1] Large in the sense of bushiness, not large as a subclass of 2^ω or ω^ω.

Bushiness

We use an analogue for infinite sets of Kumabe's notion of *goodness*, later often called *bushiness*:

Definition 5

A subset *C* of $\mathbb{N}^* = \omega^{<\omega}$ is *n-bushy* if the empty string is in *C* and every element of *C* has at least *n* many immediate extensions in *C*.

Theorem 6 (K., 2008)

There is a 3-bushy set $C \subseteq \omega^{<\omega}$ *such that (i) each infinite path* Z *through* C *is feeble; (ii)* C *is computable from* $0'$.

Proof.

Modify Slaman's $0'$-oracle construction from (Ambos-Spies, Kjos-Hanssen, Lempp, Slaman, 2004). Now we ask for sets that are so bushy that there is not just *one* acceptable path through them, but a whole *3-bushy collection* of such paths. Then the construction splits up into *subconstructions* for each of these paths. ☐

Theorem 7 (classical)

Consider a birth-death process where everyone has 3 children, each with a 50% chance of surviving and themselves having 3 children. The probability that the tribe eventually goes extinct is strictly less than 1.

Proof of Theorem 4 (Law of Feeble Subsets) from Theorem 6 (3-bushy tree of feeble sets)

Let X be a subset of $\mathbb{N}$ that is Martin-Löf random relative to $0'$. After applying an effective bijection, we may assume $X \subseteq \mathbb{N}^*$ instead.
Look at the intersection $X \cap C$. Since $C \leq_T 0'$ is 3-bushy and X is random relative to C, this is a birth-death process as in Theorem 7.
Some finite modification of X contains an infinite path through $\mathbb{N}^*$ that is contained in C.

Theorem 4 states that each 2-random set has a feeble subset.

Question 2

Does every 1-random set have a feeble subset?

Actually, we can essentially answer Question 2 if we replace the usual Bernoulli parameter $1/2$ by a *noncomputable* number p and consider μ_p-randomness.

Hippocratic randomness

Definition 8

A test for *Hippocratic* μ-randomness is a uniformly Σ^0_1 sequence $\{U_n\}_{n\in\omega}$ with $\mu(U_n) \leq 2^{-n}$.

If X passes all Hippocratic tests then X is *Hippocrates μ-random*.

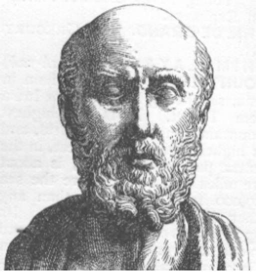

Figure: Like Hippocrates we are not consulting the oracle μ but rather looking for “natural causes”.

Proposition 9

Consider an i.i.d. sequence $Y = \{Y_n\}_{n\in\omega}$ *of Bernoulli(p) random variables, and the sample average* $\overline{Y}_n = \frac{1}{n}\sum_{i=0}^{n-1} Y_i$*. Let* $N(b) = 2^{3b-1}$ *and let*

$$U_d = \bigcup_{b\geq d}\{Y : |\overline{Y}_{N(b)} - p| \geq 2^{-b}\}.$$

Then U_d *is uniformly* $\Sigma^0_1(p)$*, and* $\mu_p(U_d) \leq 2^{-d}$*, i.e.,* $\{U_d\}_{d\in\omega}$ *is a* μ_p*-ML-test.*

The idea of the proof is to use Chebyshev's inequality and the fact that the variance of a Bernoulli(*p*) random variable is bounded (by $1/4$).

The essence of Statistics

Theorem 10

If Y is μ_p-random then $Y \geq_T p$.

Proof.

Let $\{U_d\}_{d\in\omega}$ be as in Proposition 9. Since Y is μ_p-random, $Y \notin \cap_d U_d$, so fix d with $Y \notin U_d$. Then for all $b \geq d$, we have

$$|\overline{Y}_{N(b)} - p| < 2^{-b}$$

Therefore, p is computable from Y. □

Theorem 11

There is a Hippocratic μ_p-test such that if Y passes this test then Y computes an accumulation point q of $\{\overline{Y}_n\}_{n\in\omega}$.

Proof.

Let

$$V_d := \{Y : \exists a, b \geq d \ |\overline{Y}_{N(a)} - \overline{Y}_{N(b)}| \geq 2^{-a} + 2^{-b}\}$$

Then $\{V_d\}_{d \in \omega}$ is uniformly Σ^0_1. Recall

$$U_d = \{Y : \exists b \geq d \ |\overline{Y}_{N(b)} - p| \geq 2^{-b}\}$$

We have $V_d \subseteq U_d$. [2]
Therefore $\mu_p(V_d) \leq \mu_p(U_d) \leq 2^{-d}$ for all p. Thus if Y is Hippocrates μ_p-random then $Y \notin V_d$ for some d. This allows Y to compute the limit q of the sequence $\{Y_{N(b)}\}_{b \geq d}$.

□

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[2] (If $|\overline{Y}_{N(b)} - p| < 2^{-b}$ for all $b \geq d$ then $|\overline{Y}_{N(a)} - \overline{Y}_{N(b)}| \leq |\overline{Y}_{N(a)} - p| + |p - \overline{Y}_{N(b)}| < 2^{-a} + 2^{-b}$ for all $a, b \geq d$.)

Theorem 12

If Y is Hippocrates μ_p-random then Y satisfies the Strong Law of Large Numbers for p.

Proof.

Let q_1, q_2 be rational numbers with $q_1 < p < q_2$. Let

$$W_N := \{Y : \exists n \geq N\ \overline{Y}_n \leq q_1\} \cup \{Y : \exists n \geq N\ \overline{Y}_n \geq q_2\}$$

Then $\{W_N\}_{N\in\omega}$ is uniformly Σ^0_1, and $\mu_p W_N \to 0$ effectively:

$$\mu_p = \{Y : \exists n \geq N\ \overline{Y}_n \leq q_1\} \leq \frac{1}{2|p - q_1|} \sum_{n \geq N} \frac{3}{n^2} - \frac{2}{n^3}$$

□

Theorem 13

If Y is Hippocrates μ_p-random then $Y \geq_T p$.

Proof.

By Theorem 11, Y computes the limit of a subsequence $\{Y_{N(b)}\}_{b\in\omega}$. By Theorem 12, this limit must be p. $\square$

Theorem 14

For all p, if there is a Hippocratic μ_p-test $\{U_n\}_{n\in\omega}$ such that $\{X : X \not\geq_T p\} \subseteq \cap_n U_n$, then p is computable.

Proof.

Let $\{U_n\}_{n\in\omega}$ be such a test. U_1^c has a low member X_1 and a hyperimmune-free member X_2. By assumption $X_1 \geq_T p$ and $X_2 \geq_T p$, so p is both low and hyperimmune-free, hence p is computable. □

Corollary 15

There is no universal Hippocratic μ_p-test (unless p is computable).

Turning a $\Sigma^0_1(p)$ test into a Σ^0_1 test

Definition 16

Let $\{O_n\}_{n\in\omega}$ be a universal μ_p-test for all p, i.e. $\mu_p O^p_n \leq 2^{-(n)}$ for all p and $\{(p, X, n) : X \in O^p_n\}$ is Σ^0_1.

Definition 17

Let Ψ_d denote the reduction from Theorem 11 under the assumption $Y \notin U_d$ there.

Theorem 18

If Y is Hippocrates μ_p-random then Y is μ_p-random.

Proof. We have

$$\{Y : Y \notin U_d\} \subseteq \{Y : \Psi_d^Y = p\}$$

$$\cup \{Y : Y \text{ not Hippocrates } \mu_p\text{-random.}\}$$

Let

$$V_n^{(d)} := \left\{X : \exists k \ \left(\Psi_d^X \upharpoonright k \downarrow \ \& \ X \in O_n^{\Psi_d^X \upharpoonright k}\right)\right\}$$

Then

$$V_n^{(d)} \subseteq O_n^p \cup \{Y : \Psi_d^Y \neq p\}$$

So

$$\mu_p(V_n^{(d)}) \leq \mu_p(O_n^p) + \mu_p U_d \leq 2^{-n} + 2^{-d}$$

Form the diagonal $W_n = V_n^{(n)}$; then $\{W_n\}_{n\in\omega}$ is a Hippocratic μ_p-test.
Suppose for contradiction that Y is Hippocrates μ_p-random but not μ_p-random. Then for all n, $Y \in O_n^p$. Fix d such that $Y \notin U_d$, so $\Psi_d^Y = p$. Then for all $n \geq d$, $\Psi_n^Y = p$. Then $Y \in \cap_{n\geq d} V_n^{(n)}$. So Y is not Hippocrates μ_p-random. *End of proof.*

Review: For μ_p, Hippocratic randomness is the same as (non-Hippocratic) randomness. Each μ_p-random real computes p.

Theorem 19 (Seetapun, 1995)

For each A and B, if each infinite subset and each infinite superset of A computes B, then B is recursive.

Theorem 20

If p is noncomputable, then each μ_p-ML-random real X has an infinite subset, or coinfinite superset, Y, that computes no μ_p-ML-random real.

Proof.

Each μ_p-ML-random real computes p as we have seen. By Seetaphun's Theorem there is an infinite subset, or coinfinite superset, of X that does not compute p. □

So Question 2 essentially asks whether this is true for computable numbers p, in particular $p = 1/2$.

How dense can a feeble subset $Y \subseteq X$ be?

The *principal function* of a set Y is given by $p(n) =$ the smallest element of Y that is $> p(n-1)$.

- K., 2007: principal function bounded by 4^n.
- Miller, 2008: principal function bounded by $n^{2+\varepsilon}$.

Question 3

Can a Martin-Löf random set X have a feeble subset Y of linear density, i.e. principal function bounded by $c \cdot n$?

One way to approach this question is to start with stronger reducibilities.

Definition 21

Let r be a reducibility. Y is r-feeble if Y is infinite and Y does not r-compute any ML-random set.

c, *d*, and *btt*-feeble subsets of linear density

Take Y to be a μ_p-random subset of X. Then Y has principal function bounded by $(p^{-1} + \varepsilon)n$ (Y has linear density) and Y is *c*-feeble, *d*-feeble, and *btt*-feeble.
However, using von Neumann's extractor one can show that Y is not *tt*-feeble. By using different algorithms one can show that Y is also not *l*-feeble nor *p*-feeble.

Question 4

Is there an r-feeble subset of X of linear density, for $r \in \{p, l\}$*?*
Or even for $r \in \{tt, T\}$*?*

The 7 *clones* in Post's lattice containing the constant functions correspond to reducibilities *m*, *btt*(1), *c*, *d*, *p*, *l*, *tt* and out of these 7 a μ_p-random set is feeble for exactly 4.

Reducibilities for which a μ_p-random set is feeble:

- *m*: $\{0, 1\}$
- *btt*(1): $\{0, \neg\}$
- *c*: $\{0, 1, \wedge\}$
- *d*: $\{0, 1, \vee\}$

and those for which it is not:

- *p*: $\{0, 1, \wedge, \vee\}$
- ℓ: $\{0, \leftrightarrow\}$
- *tt*: $\{\neg, \vee\}$

When is a measure “close enough” to uniform?

Theorem 22 (Kakutani 1948)

Let μ and ν be strongly positive (i.e. the p_i, q_i are bounded away from 0 and 1) generalized Bernoulli measures of parameters $\{p_i\}_{i \in \mathbb{N}}$ and $\{q_i\}_{i \in \mathbb{N}}$, respectively. If $\sum_i (p_i - q_i)^2 < \infty$ then μ and ν are equivalent (i.e. have the same null sets).

Theorem 23 (Vovk 1987)

Let μ and ν be computable strongly positive generalized Bernoulli measures of parameters $\{p_i\}_{i \in \mathbb{N}}$ and $\{q_i\}_{i \in \mathbb{N}}$, respectively. If $\sum_i (p_i - q_i)^2 < \infty$ then μMLR=νMLR.

Linear reducibility

Z is linearly reducible to Y ($Z \leq_\ell Y$) if the truth of the equation $Z(n) = 1$ can effectively be reduced to the truth of a linear equation in the variables $Y(0), Y(1), \ldots$ over the finite field $\mathbb{Z}/2\mathbb{Z} = (\{0,1\},+)$:

$$Z(43) = 1 \Leftrightarrow Y(3) + Y(5) = 0$$

$$Z(44) = 1 \Leftrightarrow Y(1) + Y(6) + Y(8) = 1$$

$$\ldots$$

One way to state this:

Definition 24

$Z \leq_\ell Y$ if there is a recursive function f such that for all n,

$$n \in Z \leftrightarrow |D_{f(n)} \cap Y| \equiv 1 (\text{mod } 2).$$

An improved von Neumann extractor (in a sense)

Theorem 25

A μ_p-random set is not ℓ-feeble.

Proof.

Suppose Y is μ_p-random. The μ_p-probability that $|D_{f(n)} \cap Y| \equiv 1 (\text{mod } 2)$ is computable from p and converges effectively to $1/2$ as $|D_{f(n)}| \to \infty$. [3]
Let $D_{f(n)}$ be disjoint finite sets that are so large that if $p_i := \mu_p(\{Y : |D_{f(i)} \cap Y| \equiv 1 (\text{mod } 2)\})$ then $\sum_i (p_i - \frac{1}{2})^2 < \infty$.
If Z is reduced to Y via $D_{f(n)}$, then Z is ML-random. $\square$

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[3] The reason is that if $D_{f(n)}$ is very large, then it is likely to contain a large amount of numbers from Y, and this amount is about equally likely to be odd or even.

Positive reducibility

Definition 26

$Z \leq_p Y$ iff there is a recursive function f such that for all n, $n \in Z$ iff $\exists i \leq k_i \; D_{f(n,i)} \subseteq Y$.

We can rewrite this:

$$\neg \forall i \leq k_i [D_{f(n,i)} \not\subseteq Y] \qquad (*)$$

If the $D_{f(n,i)}$ are disjoint then the events $[D_{f(n,i)} \not\subseteq Y]$ are independent and so we can calculate the probability of (*). By taking $|D_{f(n,i)}|$ large enough and taking k_i large enough we can get this probability as close to $1/2$ as desired. Thus,

Theorem 27 (A second improved von Neumann extractor)

Let $0 < q < 1$. *A* μ_q*-random set is not p-feeble.*

Ultimately, we would like to know if there are feeble subsets that are almost random themselves. Thus:

Question 5

Is there a ML-random X and a Schnorr random $Y \subseteq X$ such that Y is feeble?

Similar question: Is there a ML-random X and Schnorr random $Y \subseteq X$ such that Y is not ML-random?

Not every Schnorr random set is contained in some Martin-Löf random set; consider a Schnorr random set that does not have DNC degree.

A better version of this question:

Question 6

Is it true that for each Schnorr random set X and any finite collection of Schnorr tests $\mathcal{U} = \{U^{(1)}, \ldots, U^{(m)}\}$, there is a subset $Y \subseteq X$ that computes no Schnorr random but passes all the tests in $\mathcal{U}$?

The End