Gaps for 2-random reals

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1-randoms have excess complexity

A is 1-random…

- iff $K(A \upharpoonright n) \geqslant^+ n$ (Levin; Schnorr)
- iff $K(A \upharpoonright n) - n \to \infty$ (Chaitin)
- iff $\sum_{n \in \omega} 2^{n - K(A \upharpoonright n)} < \infty$ (Ample Excess: M. and Yu).

Any real with initial segment complexity greater than n, has complexity somewhat significantly greater than n.

We know a lot about the *excess complexity*.

There is no gap

There is no single gap that works for all 1-random reals.

Theorem (Downey and Bienvenu; M. and Yu)

If $f\colon \omega \to \omega$ is *any* unbounded function, then there is a 1-random A such that $(\exists^\infty n)\ K(A \upharpoonright n) < n + f(n)$.

(Downey and Bienvenu proved this under the assumption that f is monotonic.)

The proof requires that we exert fairly fine control over dips in $K(A \upharpoonright n) - n$. For that we use *the bounding lemma*.

Bounding Lemma (M. and Yu)

If $\sum_{n\in\omega} 2^{-g(n)} < \infty$ and $g \leqslant_T A$ with use n, then $K(A \upharpoonright n) \leqslant^+ n + g(n)$.

Using $g \upharpoonright (n+1)$ we could give every string of length n a description of length $n + g(n) + O(1)$ (Kraft–Chaitin).

Let σ be the description of $A \upharpoonright n$. So σ codes $A \upharpoonright n$, from which we can compute $g \upharpoonright (n+1)$, from which we can decode σ. We would know how to read the message if only we knew what the message said. The heart of the proof is resolving this circularity.

Ample excess is tight

Theorem

If $\sum_{n\in\omega} 2^{-f(n)} < \infty$, then there is a 1-random $A \in 2^\omega$ such that

$$K(A \upharpoonright n) \leqslant^+ n + f(n).$$

Proof Idea.

Find a function g such that

1. $g \leqslant f$,
2. $\sum_{n\in\omega} 2^{-g(n)} < \infty$,
3. g can be coded in a compact way.

Use (Kučera–)Gács to find a 1-random A such that $g \leqslant_T A$ with use n. Apply the bounding lemma. □

There is no gap

Theorem (Downey and Bienvenu; M. and Yu)

If $f\colon \omega \to \omega$ is *any* unbounded function, then there is a 1-random A such that $(\exists^\infty n)\ K(A \upharpoonright n) < n + f(n)$.

Proof.

Let g be any function such that

1. $\sum_{n\in\omega} 2^{-g(n)} < \infty$,
2. $\limsup_{n\in\omega} f(n) - g(n) = \infty$.

Apply the previous theorem. □

Review

- A 1-random implies $K(A \upharpoonright n) - n \to \infty$.
- No lower bound on this divergence for all 1-randoms.

Gaps for stronger randomness notions?

An easy consequence of the no-gap theorem:

Corollary

If $f\colon \omega \to \omega$ is *any* unbounded function, then there is a weak 2-random A such that $(\exists^\infty n)\ K(A \upharpoonright n) < n + f(n)$.

On the other hand:

Proposition

There is a monotonic f such that if A is Schnorr 2-random, then $K(A \upharpoonright n) \geqslant^+ n + f(n)$.

We will focus on gaps for 2-random reals.

2-random reals

Solovay first proved the existence of a complexity gap for 2-randomness. He also showed that Ω did not respect this gap. More on Solovay soon…

Recall

The halting probability Ω is Turing equivalent to $\emptyset'$.

So, A is 2-random…

- iff A is $\emptyset'$-random
- iff A is Ω-random
- iff Ω is A-random (Van Lambalgen's theorem)

A gap characterizing 2-random reals

Theorem (M. and Yu)

A and B are random relative to each other iff

$$K(A \upharpoonright n) + C(B \upharpoonright n) \geqslant^{+} 2n.$$

Therefore, A is 2-random if and only if

$$K(A \upharpoonright n) \geqslant^{+} n + f(n),$$

where $f(n) = n - C(\Omega \upharpoonright n)$.

What's left?

- Characterize 2-randomness with a *monotonic* gap.
- Understand Solovay's work on this subject.

Slow growing functions

Solovay considered two "inverse busy beaver" functions:

$$\begin{aligned} \alpha(n) &= \min_{m \geqslant n} K(m), \\ s(n) &= -\log\left(\sum_{m \geqslant n} 2^{-K(m)}\right). \end{aligned}$$

Both are nondecreasing, unbounded and $\emptyset'$-computable.

Theorem (Solovay)

- $s(n) \leqslant^{+} \alpha(n)$,
- $\alpha(n) \leqslant^{+} s(n) + K(s(n))$.

Solovay's gap for 2-randoms

Theorem (Solovay)

- $K^{\emptyset'}(n) \leqslant K(n) - \alpha(n) + O(\log \alpha(n))$,
- There is a c such that if A is 2-random, then
$$K(A \upharpoonright n) \geqslant n + \alpha(n) - c \log \alpha(n).$$

We will show that:

Theorem

- There is no A for which $K(A \upharpoonright n) \geqslant^+ n + \alpha(n)$,
- A is 2-random if and only if $K(A \upharpoonright n) \geqslant^+ n + s(n)$.

We have a natural monotonic gap characterizing 2-randomness.

Why α is not a gap

Theorem (M. and Yu)

The following are equivalent for g:

1. $\sum_{n \in \omega} 2^{g(n) - K(n, g(n))}$ diverges,
2. For almost every A, it is not the case that $K(A \upharpoonright n) \geqslant^+ n + g(n)$.

Take m such that $\alpha(m) = K(m)$.

Then $K(m, \alpha(m)) = K(m, K(m)) =^+ K(m) = \alpha(m)$, so $2^{\alpha(m) - K(m, \alpha(m))}$ is bounded below for such m. Therefore, $\sum_{n \in \omega} 2^{\alpha(n) - K(n, \alpha(n))}$ diverges.

Therefore, $K(A \upharpoonright n) \not\geqslant^+ n + \alpha(n)$, for almost every A.

Why α is not a gap (cont.)

From:

- $K(A \restriction n) \not\geqslant^{+} n + \alpha(n)$, for almost every A, and
- $\alpha \leqslant_T \emptyset'$,

it follows that $K(A \restriction n) \not\geqslant^{+} n + \alpha(n)$ when A is 2-random.

But $K(A \restriction n) \geqslant^{+} n + \alpha(n)$ implies $K(A \restriction n) \geqslant^{+} n + s(n)$ (since $s \leqslant^{+} \alpha$), which implies that A is 2-random (see below).

Corollary

There is no A for which $K(A \restriction n) \geqslant^{+} n + \alpha(n)$.

A characterization of s

Let $\{\Omega_n\}_{n \in \omega}$ be a computable, non-decreasing sequence of rational numbers converging to Ω.

Definition

$\gamma(n) = -\log(\Omega - \Omega_n)$.

Proposition

The definition of γ is independent, up to a constant, of the choice of Ω and $\{\Omega_n\}_{n \in \omega}$.

This follows from Kučera and Slaman, who show that *every* computable, monotonic sequence approximating a 1-random c.e. real has essentially the same rate of convergence.

A characterization of s (cont.)

$$s(n) = -\log\left(\sum_{m \geqslant n} 2^{-K(m)}\right),$$
$$\gamma(n) = -\log(\Omega - \Omega_n).$$

Proposition

$\gamma(n) =^+ s(n)$.

Proof.

Take $\Omega = \sum_{m \in \omega} 2^{-K(m)}$ and $\Omega_n = \sum_{m<n} 2^{-K_n(m)}$.

Then clearly $\Omega - \Omega_n \geqslant \sum_{m \geqslant n} 2^{-K(m)}$, so $\gamma(n) \leqslant^+ s(n)$.

But $K(m) \leqslant^+ -\log(\Omega_{m+1} - \Omega_m)$, so $s(n) \leqslant^+ \gamma(n)$. □

Another slow growing function

We use another slow growing function and close relative of γ.

Definition

$\widehat{\gamma}(n) = \max\{k \colon \Omega_n \upharpoonright k = \Omega \upharpoonright k\}$.

Notes

- Since $\Omega - \Omega_n \leqslant 2^{-\widehat{\gamma}(n)}$, we have $\widehat{\gamma} \leqslant^+ \gamma$.
- It is not clear that $\widehat{\gamma}$ is independent of the choice of Ω and $\{\Omega_n\}_{n \in \omega}$.
- Also not proved: is $\widehat{\gamma}$ different from γ?

Another gap for 2-random reals

Theorem

The following are equivalent for $A \in 2^\omega$:

1. A is 2-random,
2. $K(A \upharpoonright n) \geqslant^+ n + \gamma(n)$.

The same holds with γ replaced by $\widehat{\gamma}$.

Proof.

We may assume that A is 1-random, otherwise neither condition holds.

(2) $\Longrightarrow$ (1) for $\widehat{\gamma}$: Define g as follows. To find $g(n)$, look for the the first *unused* $\sigma \in 2^{<\omega}$ such that $U_n^{A \upharpoonright n}(\sigma) = \tau$ and $\Omega_n \upharpoonright |\tau| = \tau$. Mark σ as *used* and set $g(n) = |\sigma|$. Otherwise, let $g(n) = n$.

Proof (Cont.)

Note that $g(n)$ can be computed from $A \upharpoonright n$. Also note that $\sum_{n\in\omega} 2^{-g(n)} < \infty$. Therefore, by the bounding lemma, there is a $d \in \omega$ such that $K(A \upharpoonright n) \leqslant n + g(n) + d$.

If Ω is not A-random, fix c and find an m such that $K^A(\Omega \upharpoonright m) \leqslant m - c$. Let σ be a minimal U^A-program for $\Omega \upharpoonright m$. This σ will eventually be used in the definition of g for some n. Thus $g(n) = |\sigma| \leqslant m - c$. But $\Omega_n \upharpoonright m = \Omega \upharpoonright m$, so $\widehat{\gamma}(n) \geqslant m$. Therefore

$$K(A \upharpoonright n) \leqslant n + g(n) + d \leqslant n + m - c + d \leqslant n + \widehat{\gamma}(n) - c + d.$$

For all c, there is such an n, hence $K(A \upharpoonright n) \not\geqslant^+ n + \widehat{\gamma}(n)$.

(1) $\Longrightarrow$ (2) for γ: Use the ample excess lemma. $\square$

Review

Review

- A 1-random implies $K(A \upharpoonright n) - n \to \infty$.
- No lower bound on this divergence works for all weak 2-random reals.
- There is a gap for all Schnorr 2-random reals.
- $\alpha(n) = \min_{m \geqslant n} K(m)$ is not a gap for any 1-random.
- There is a monotonic gap $\gamma(n) =^{+} s(n)$ *characterizing* 2-randomness.

Final note

It is not the case that $K^{\emptyset'}(n) \leqslant^{+} K(n) - \gamma(n)$.

This is because $\sum_{n \in \omega} 2^{-K(n)+\gamma(n)}$ diverges.

Thank You