Effective packing dimension and traceability

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Notation

- Real is a member of Cantor space 2^{ω} with topology with basic clopen sets $[\sigma] = \{\sigma\alpha : \alpha \in 2^{\omega}\}$ whose measure is $\mu([\sigma]) = 2^{-|\sigma|}$.
- Strings = members of $2^{<\omega} = \{0, 1\}^{*}$.

Martingales and supermartingales

- Recall that a martingale is a betting strategy $F : 2^{<\omega} \to \mathbb{R}^+ \cup \{0\}$ so that $F(\sigma) = \frac{F(\sigma 0)+F(\sigma 1)}{2}$. If $=$ is replaced by $\leq$ then this is a supermartingale.
- A real α succeeds if $\limsup_{n\to\infty} F(\alpha \upharpoonright n) = \infty$.
- Recall α is 1-random iff no c.e. supermartingale succeeds on alpha. Here c.e. is computable from below. (Schnorr)

Hausdorff dimension

- Borel (late 19th century), Lebesgue (1904) measure.
- In any *n*-dimensional Euclidean space, Carathéodory 1914

  $$\mu^s(A) = \inf\{\sum_i |I_i|^s : A \subset \cup_i I_i\},$$

  where each I_i is an interval in the space.
- 1919 Hausdorff *s* fractional; and refine measure 0.
- Vary the way we measure open set (by factor *s*):
  for $0 \leq s \leq 1$, the *s*-measure of a clopen set $[\sigma]$ is

  $$\mu_s([\sigma]) = 2^{-s|\sigma|}.$$

- The classical Hausdorff dimension of a set of reals is the *s* which gives a “jump”.

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- The effective versions of the classical notions of fractional dimensions, via martingales.
- (Lutz) An s-gale is a function $F : 2^{<\omega} \mapsto \mathbb{R}$ such that

  $$F(\sigma) = 2^{-s}(F(\sigma 0) + F(\sigma 1)).$$

  Inflation!
- Theorem For a class X the following are equivalent:
  - (i) $\dim(X) = s$.
  - (ii) $s = \inf\{s \in \mathbb{Q} : X \subseteq S[d]$ for some s-gale $d\}$.
- Lutz: the dimension is the most hostile environment (i.e. most unfavorable payoff schedule, the smallest s) in which a single betting strategy can achieve infinite winnings.

- C.e. martingales give effective version of Hausdorff dimension.
- Remarkably,
  Theorem (Mayordomo): The effective Hausdorff dimension of a real α is

$$\liminf_{n\to\infty} \frac{K(\alpha \upharpoonright n)}{n} = (\liminf_{n\to\infty} \frac{C(\alpha \upharpoonright n)}{n})$$

Extracting randomness

- An easy example of something which has effective dimension $\frac{1}{2}$ is to take Ω and spread it out by inserting 0's every second bit.
  Is this the only way?
- Question: (Reimann, Terwijn) Can randomness always be extracted from positive dimension?
  What about higher dimension?

Dimension extraction

- Theorem (Miller) There is a Turing cone of dimension $\frac{1}{2}$ (broken dimension).
- Theorem (Zimand) Hausdorff dimension 1 can be extracted from two independent sources of positive dimension.
- Theorem (Greenberg and Miller) There is a real of effective Hausdorff dimension 1 of minimal degree (hence computes no random).

The Greenberg-Miller Theorem

- (GM) There is a real of effective Hausdorff dimension 1 and of minimal degree.
- The proof idea is to use DNR functions. What can they uniformly compute?
- Modify the Kumabe-Lewis construction.

Packing dimension

- Idea is to replace outer measure by inner measure. Look for a dense packing.
- Classically we can look at the packing dimension in a similar way.
- Athreya, Hitchcock, Lutz, Mayordomo also characterized packing dimension in terms of martingales. Strongly succeed.
- Can define the effective packing dimension. This is characterized as

$$\limsup_{n\to\infty} \frac{K(\alpha \upharpoonright n)}{n} \left(= \limsup_{n\to\infty} \frac{C(\alpha \upharpoonright n)}{n}\right).$$

- Interesting as 2-generics have high efffective packing dimension. The set
$$X_k = \{\sigma \in 2^{<\omega} : K(\sigma) > (1 - \frac{1}{k})|\sigma|\}$$
is computable in $\emptyset'$. Measure meets category (co-meagre).
- Fundamental question: what Turing degrees contain reals of high packing dimension?

Theorem (Fortnow, Hitchcock, Aduri, Vinochandran, Wang)

If α has packing dimension > 0, then $\exists \beta \equiv_{wtt} \alpha$ of packing dimension $1 - \varepsilon$.

- Hence for degrees a 0-1 Law for effective packing dimension (no broken dimension).
- Open Question: is there a real of effective packing dimension 1 inside each degree of packing dimension 1?

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How to work with packing dimension

The following lemma is implicit in, e.g. Conidis

Lemma

There is a computable mapping $(\sigma, \varepsilon) \mapsto n_\varepsilon(\sigma)$ *which maps a finite binary string* $\sigma \in 2^{<\omega}$ *and a positive rational* ε *to a natural number* n *such that there is some binary string* τ *of length* n *such that*

$$\frac{K(\sigma\tau)}{|\sigma\tau|} \geq 1 - \varepsilon.$$

A minimal degree of packing dimension 1

Theorem (Downey, Greenberg)

There is a minimal degree of packing dimension 1.

- We force with clumpy trees. These are clumps generated by the n_ε above and separated by long stretches.
  - At odd stages, force minimality Spector style.
  - At even stages pick an extension within the clump of high complexity.
- At stage 0, start with the full tree $2^{<\omega}$ (clearly clumpy).
- At an odd stage s, given a clumpy tree T.

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- First of all, ask if we can always force splits

$$(\forall\sigma \in T)(\exists x, \tau_1, \tau_2 \in T)(\tau_1, \tau_2 \supset \sigma \ \wedge \ \Phi^{\tau_1}(x) \neq \Phi^{\tau_2}(x)).$$

- If no, take full subtree. Otherwise define a clumpily-splitting subtree $S \subset T$. Suppose $\sigma \in T$, and is the stem of a clump. Let $X = \{\sigma\tau : |\tau| = n_\varepsilon(\sigma)\}$.
- It is not enough to find splits above each $\sigma\tau \in X$ (on a single input x).
- For each $\sigma\tau \in X$ find $\gamma \supset \sigma\tau$ such that all Φ^γ are incomparable.
- The same kind of idea can be used to construct a rank one c.e. real of packing dimension 1. (Conidis)
- Have a clump, move only left, with long stretches of zeroes extending.

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Degrees

- (Downey, Jockusch, Stob) Recall that **a** is array noncomputable iff for all $f \leq_{wtt} \emptyset'$ there is a function $g \leq_T \mathbf{a}$ such that
$$\exists^\infty n(g(n) > f(n)).$$
- Array computability is stronger than (the uniform version of) being totally ω-c.e.
(DG) **b** is totally ω-c.e. iff all functions $g \leq_T \mathbf{b}$ are ω-c.e..
- (Downey, Greenberg, Weber) These c.e. degrees are definable via critical triples.

- For these two classes, methods of multiple permitting (and anti-permitting) were developed. They show up in randomness.
- (Brodhead, Downey, Ng) computable finite randomness.
- (Barmpalias, Downey, Greenberg) common upperbounds in the ibT- and cL-degrees.

Theorem (Downey, Greenberg)

A c.e. degree contains a real of effective packing dimension 1 iff it is array noncomputable.

- ($\Rightarrow$): recall that for c.e. sets, array computable is the same as c.e. traceable. (Ismukhametov)
- A is c.e. traceable, if for any computable order h, and all functions $g \leq_T A$, there is a weak array $\{W_{q(n)} : n \in \omega\}$, such that $|W_{q(n)}| < h(n)$ and $g(n) \in W_{q(n)}$.
- Suppose A is c.e. traceable (not necessarily a c.e. set), use $g(n) = A \upharpoonright n$.
- For every order function h, $C(A \upharpoonright n) \leq \log n + h(n) + O(1)$. This is an idea of Kummer.

- ($\Leftarrow$): suppose A is anc. To make the real $\alpha \leq_T A$ complicated, at some clump we need to be able to move left often enough lift the dimension.
- α requires “multiple permitting" from A. Use the classical definition of anc.
- The c.e. set A is anc iff for all very strong arrays $\{D_{k(n)} : n \in \omega\}$ (i.e. $|D_{k(n+1)}| > |D_{k(n)}|$), for all e there is a n with $W_e \cap D_{k(n)} = A \cap D_{k(n)}$. This is a kind of “multiple permitting”.

Related results, Kummer's gap

- We know that a c.e. set can have maximal complexity $C(A \upharpoonright n)$ as $2 \log n$. Solovay showed that it is impossible to have that almost always.
- (Kummer) Either a c.e. degree is array computable and all initial segments are within $(1+\varepsilon)\log n + O(1)$

  or

  the degree contains a set which is infinitely often $2 \log n - O(1)$.

Characterization

- What are the degrees with packing dimension 1?
  This is completely classified for the c.e. degrees.
- Outside the c.e. degrees, every array noncomputable degree also computes a set of packing dimension 1.
- Proof sketch: every anc degree bounds a pb-generic real. α is pb-generic if it meets every pb-dense set of strings. A set of strings S is pb-dense if there is an f such that for every σ, $\sigma \subseteq f(\sigma)$ and $f(\sigma) \in S$. Here f is effectively approximable with primitive recursive number of mind changes.
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Characterization

- First guess: ***a*** has packing dimension 1 iff ***a*** is anc.
- Unfortunately ($\Rightarrow$) fails: superlow randoms are array computable, and similarly hypermmune-free randoms.
- Second guess: ***a*** has packing dimension 1 iff ***a*** is not c.e. traceable. (random reals are all non-c.e. traceable).

Theorem (Downey and Ng)

There is a Δ^0_3 real A which is of hyperimmune-free degree and not c.e. traceable, such that every real $\alpha \leq_T A$ has effective packing dimension 0.

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Proof of the HIF case

- Both proofs again use this notion of highly branching trees instead of Cantor space.
- To make A HIF we use a finite extension argument over a Spector-style forcing. However the standard construction makes it computably traceable.
- Make A HIF but not c.e. traceable (Terwijn's thesis). The forcing conditions here are lumpy trees.
- $T : \omega^{<\omega} \mapsto \omega^{<\omega}$ is g-lumpy, if for every σ, $T(\sigma)$ has at least $g(|T(\sigma)|)$ many successors on T.

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Proof of the HIF case

- Start with a highly branching tree T_0, each node at level n have $n+1$ immediate successors. This is clearly g-lumpy for $g := id$.
- Ask the usual question. If we can always force convergence for Φ everywhere, build an id-lumpy subtree $S \subset T_0$.
- We can get a computable bound for Φ, while killing all c.e. traces with cardinality bounded by id.

Proof of the HIF case

- We also need to satisfy requirement of the form

  $$\mathcal{N} : C(\Phi^A \upharpoonright n) \leq \frac{n}{2} \text{ for every } n.$$

  This is enough to ensure packing dimension of Φ^A is 0.
- Suppose we are given a g-lumpy tree T. We want to find an *id*-lumpy $S \subset T$, such that $\mathcal{N}$ holds for every $X \in [S]$.
- Problem: suppose $\tau \in T$ and $\Phi^\tau \downarrow$. No general relationship between $|\tau| \gg |\Phi^\tau|$.
- S has to be at least *id*-lumpy. We may have to describe $\Phi^{\tau \frown i}$ for ($|\tau|$ many i's). This corresponds to $2^{-|\Phi^\tau|}$ for each $i \leq |\tau|$.
- Can we choose a longer $\tau \in T$ so that Φ^τ cost less to describe?

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Majority vote argument

- Solution: restrict the forcing conditions T to be very lumpy trees, i.e. g-lumpy for some computable $g \gg id$.
- Use the fact that we are describing Φ^X, which are *sets* rather than *functions*. This allows to carry out “majority vote” argument.
- Start with a level x_1 (first lump on S). Pick $L(x_1)$ large, and let $g(x_1) = x_1 2^{L(x_1)}$. Accounting for Φ^X, $X \in [S]$:
  - 1 piece of length $L(x_1)$,
  - x_1 pieces of length $L(x_2)$,
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Majority vote argument

- Given g-lumpy tree T, we produced a id-lumpy $S \subset T$ for one requirement.
- Combining all the requirements: Generally when we have a g_i-lumpy T_i, we produce a g_{i+1}-lumpy subtree T_{i+1}.
- Choose a fast-growing g_0 to begin, such that every subsequent $g_i \gg id$.
- The majority vote argument also works in making A Δ^0_2 instead of HIF.
- Visit every successor of a lump and get a majority vote. Once a majority vote is decided, never move again unless told to do so by diagonalization.

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Further thoughts

- The obvious characterizations for degrees of packing dimension 1 do not work.
- By $0-1$ law, "strong" lowness property... maybe this has something to do with lowness like Schnorr, Kurtz etc.
- The former is related to c.e. traceable, latter are HIF and not DNR.
- Question: does every DNR function compute a real of packing dimension 1?
  Analogue of Greenberg, Miller.

The End

- Thank you.