

# New directions in computability and randomness

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# Some areas of possible future research

## The complexity of sets:

- Partial relativization and weak reducibilities
- Cost functions
- Classes related to the  $K$ -trivials

## Randomness:

- Randomness notions between 2-random and 1-random (such as Demuth random, or Schnorr random relative to  $\emptyset'$ )
- Randomness via computational complexity theory
- Randomness via higher descriptive set theory

**Both:** Randomness/lowness with infinite objects other than subsets of  $\mathbb{N}$  (such as continuous functions). Work of Asarin/Prokovskii, Fouché, Kjos-Hanssen.

## Part 1

# Partial relativization and weak reducibilities

We want to partially relativize a class  $\mathcal{C}$  in such a way that “ $A \in \mathcal{C}^B$ ” is transitive.

# Partial relativization

Let  $\mathcal{C} \subseteq 2^{\mathbb{N}}$  be a relativizable class (for instance, a lowness property, saying that a set is close to being computable).

Usually the relation “ $A \in \mathcal{C}^B$ ” is **not** transitive.

For instance, if  $\mathcal{C}$  is the usual lowness  $A' \leq_T \emptyset'$ , we have

$$\mathcal{C}^B = \{A: (A \oplus B)' \leq_T B'\}.$$

Take low sets  $A, D$  such that  $\emptyset' \equiv_T A \oplus D$ , then

$$A \in \mathcal{C}^{\emptyset}, \emptyset \in \mathcal{C}^D, \text{ but } A \notin \mathcal{C}^D.$$

To obtain transitivity, we relativize  $\mathcal{C}$  to  $B$  only **partially**.

We will say that  $A$  has property  $\mathcal{C}$  **by**  $B$ , or **plop**  $B$ .

For instance, if  $\mathcal{C}$  is lowness we have

$$\mathcal{C}^{by B} = \{A: A' \leq_T B'\}.$$

Now, the relation “ $A \in \mathcal{C}^{by B}$ ” is transitive.

### Dictionary

plop |pläp|

noun

a short sound as of a small, solid object dropping into water without a splash.

verb ( **plopped** , **plop·ping** )

fall or cause to fall with such a sound : [ intrans. ] *the stone plopped into the pond* [ trans. ] | *she plopped a sugar cube into the cup.*

- ( **plop oneself down** ) sit or lie down gently but clumsily : *he plopped himself down on the nearest chair.*

ORIGIN early 19th cent.: imitative.

# Definition of weak reducibility

- The idea is that  $A \leq_T B$  is too **strong**, because  $B$  knows everything about  $A$ .
- In case of  $A \leq_W B$ , the set  $B$  only knows a certain aspect of what  $A$  can do.

## Definition

A preordering  $\leq_W$  on  $2^{\mathbb{N}}$  is called **weak reducibility** if

- $\leq_W$  as a relation on sets is  $\Sigma_n^0$  for some  $n$  ;
- $A \leq_T B$  implies  $A \leq_W B$ ;
- $X' \not\leq_W X$  for each set  $X$ .

# Examples of weak reducibilities

- $A \leq_{\text{cdom}} B \Leftrightarrow$   
each  $A$ -computable function is dominated by a  $B$ -computable function.  
 $B$  knows how quickly the functions computed by  $A$  grow.
- $A \leq_{LK} B \Leftrightarrow \forall x [K^B(x) \leq^+ K^A(x)].$   
 $B$  can emulate the compression power of  $A$ .

# How a relativizable class yields weak reducibilities

Given a relativizable class  $\mathcal{C}$ , there are two ways to obtain a weak reducibility  $A \leq_W B$ :

(i)  $A \leq_W B \Leftrightarrow A \in \mathcal{C}^{by B}$ ,

for the “right” type of partial relativization. (Usually there is at most one way to do it right.) For instance, if  $\mathcal{C} = \text{Low}$ , we obtain  $A' \leq_T B'$ .

(ii)  $A \leq_W B \Leftrightarrow \mathcal{C}^A \subseteq \mathcal{C}^B$ .

For instance, if  $\mathcal{C} = \Delta_2^0$ , we obtain  $A' \leq_T B'$ .  
(We apply (i) to the class  $\{X: \mathcal{C}^X \subseteq \mathcal{C}\}$ .)

# Associated lowness and highness properties

For each weak reducibility  $\leq_W$  we have

- a lowness property  $Z \leq_W \emptyset$  (“almost computable”);
- a highness properties  $\emptyset' \leq_W Z$  (“almost above  $\emptyset'$ ”).

They are disjoint because  $X' \not\leq_W X$  for each set  $X$ .

A further example of a weak reducibility, due to Nies (2005), is

$$A \leq_{LR} B \Leftrightarrow \text{each } B\text{-random set is } A\text{-random.}$$

- The associated lowness property is being **low for random**.
- The highness property is equivalent to being **uniformly almost everywhere dominating**, by Kjos-Hanssen, Miller, Solomon (to appear).

# Jump traceability

- An **order function** is a nondecreasing, unbounded, computable function.
- $J^X$  denotes the universal partial computable functional with oracle  $X$ .
- A **computably enumerable trace** is a uniformly computably enumerable sequence  $(T_x)_{x \in \mathbb{N}}$  of finite sets. It is a trace for a partial function  $\psi$  if  $\psi(x) \downarrow$  implies  $\psi(x) \in T_x$ .
- $A$  is called **jump traceable** if there is a c.e. trace  $(T_e)_{e \in \mathbb{N}}$  for  $J^A$ , and an order function  $h$  such that  $|T_e| \leq h(e)$  for each  $e$ .

# Plopping jump traceability

## Definition (Simpson, 2006)

$A$  is **jump traceable by (or plop)**  $B$ , written  $A \leq_{JT} B$ , if there is

- a c.e. trace  $(T_e)_{e \in \mathbb{N}}$  relative to  $B$  for  $J^A$ , and
- an order function  $h$

such that  $|T_e| \leq h(e)$  for each  $e$ .

In contrast, to define jump traceable **relative** to  $B$ , one would require the existence of a  $B$ -c.e. trace for  $J^{A \oplus B}$  instead of  $J^A$ , but the bound for this trace need only be computable in  $B$ .

The rules of thumb for successfully plopping to an oracle  $B$  a property of a set  $A$ :

- write  $A$  instead of  $A \oplus B$ ;
- don't relativize computable bounds.

# The weak reducibility $\leq_{JT}$

It is not hard to show that  $\leq_{JT}$  is a  $\Sigma_3^0$  relation on sets. Further,  $A \leq_T B$  implies  $A \leq_{JT} B$ , and  $A' \not\leq_{JT} A$ .

## Proposition

*The relation  $\leq_{JT}$  is transitive.*

# Which theorems/proofs survive a partial relativization?

## Theorem (Nies 05)

*A is low for ML-randomness  $\Leftrightarrow$  A is low for prefix-free complexity K, i.e.,  $\forall x [K(x) \leq^+ K^A(x)]$ .*

This becomes:  $\leq_{LR}$  is equivalent to  $\leq_{LK}$ ,  
by Kjos, Miller, Solomon (to appear). Compared to the theorem  
above, a different proof is needed.

## Theorem (Figueira, N, Stephan 07)

*Let C be plain descriptive string complexity. Then  
 $A \leq_{JT} \emptyset \Leftrightarrow \forall x [C(x) \leq^+ C^A(x) + h(C^A(x))]$  for some order  
function h.*

Plopping the proof, this becomes:

$$A \leq_{JT} B \Leftrightarrow$$

$\forall x [C^B(x) \leq^+ C^A(x) + h(C^A(x))]$  for some order function h.

# A theorem that cannot be plopped

## Theorem (Nies 2002)

Let  $A$  be c.e. Then

$A$  is jump traceable  $\Leftrightarrow A$  is superlow (i.e.,  $A' \leq_{tt} \emptyset'$ ).

Let  $A = \emptyset'$  and try to plop this theorem to a set  $B$ . We have

$$\emptyset' \leq_{JT} B \Rightarrow \emptyset'' \leq_{tt} B' \text{ (} B \text{ is superhigh)}$$

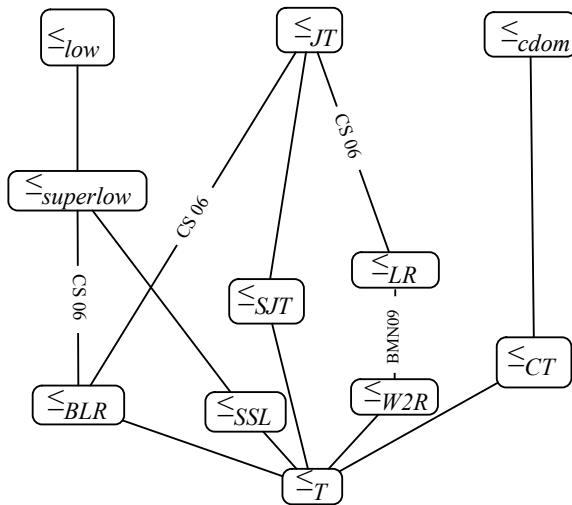
by a result of Simpson.

The converse direction, however, **fails**: there is a superhigh jump traceable set  $B$  (Kjos-H and Nies). Then  $B \not\leq_{JT} \emptyset'$ .

# Table of some weak reducibilities

Weak reducibility	Lowness property	Highness prop.
$\leq_T$	computable	$\geq_T \emptyset'$
$\leq_{LR} \Leftrightarrow \leq_{LK}$	Low(MLR) = low for $K$	u.a.e.d
$\leq_{JT}$	jump traceable	$\geq_{JT} \emptyset'$
$A' \leq_{tt} B'$	superlow	superhigh
$A' \leq_T B'$	low	high
$\leq_{CT}$	comp. traceable	$\geq_T \emptyset'$
$\leq_{\text{cdom}}$	comp. dominated	$\geq_T \emptyset'$
$\leq_{BLR}$ (Cole & Simpson)	jump tr. & superlow	$\geq_{JT} \emptyset'$

# Diagram of weak reducibilities



We say that  $A$  is a **base for**  $\leq_W$  if  $A \leq_W Z$  for some  $Z$  that is ML-random relative to  $A$ . Let  $\leq_{SJT}$ ,  $\leq_{CT}$  be the weak reducibilities obtained by plopping strong jump traceability, and computable traceability, respectively.

- (Ng) For both  $\leq_{SJT}$  and  $\leq_{CT}$ , the cone below  $\emptyset'$  has size continuum.
- (Ng) The only bases for  $\leq_{JT}$  are the jump traceable sets.
- (Barmpalias, Lewis, Stephan) Some base for  $\leq_{LR}$  is not low for ML-randomness.

# Directions of study for weak reducibilities

- **Degree theoretic questions:** existence of minimal degrees, of minimal pairs, of nontrivial suprema.
- **The cardinality of single degrees, and lower cones.** For  $\leq_{LR}$ , we know that each degree is countable; in fact  $A \equiv_{LR} B \Rightarrow A' \equiv_{tt} B'$ . (Nies, 2005 showed this for  $\leq_{LK}$ . Now use Kjos/Miller/Solomon that  $\leq_{LK} \Leftrightarrow \leq_{LR}$ .)  
The  $LR$  lower cone below  $\emptyset'$  (and in fact below each non- $GL_2$ ) is uncountable (Barnmpalias, Lewis, Soskova).
- **Implications between weak reducibilities.** For instance, does  $\leq_{SJT}$  imply  $\leq_{LR}$ ? Is  $A \leq_{CT} B \Leftrightarrow SR^B \subseteq SR^A$ ?
- **Theory of Borel equivalences.** For instance, is  $\equiv_{LR}$  Borel complete for countable Borel equivalence relations?

# Some references for Part 1

- Barmpalias et al. papers on  $\leq_{LR}$
- G. Barmpalias, J. Miller, A. Nies. **Randomness notions and partial relativization**. To appear.
- Selwyn Ng, thesis and papers.
- A. Nies. **Computability and randomness**, Oxford, 2009.  
Section 8.4

## Part 2

# Cost functions

# What cost functions are good for

Cost functions are used to analyze certain classes of  $\Delta_2^0$  sets. Mostly, these classes are lowness properties (saying the set is close to computable), such as being  $K$ -trivial, or strongly jump traceable.

Cost functions prove the following results:

- Each  $K$ -trivial set is Turing below a **c.e.**  $K$ -trivial set (Nies).
- Each null  $\Sigma_3^0$  class of ML-random sets has a simple Turing lower bound. Moreover, this lower bound is obtained via an injury-free construction (Hirschfeldt, Miller).
- If a c.e. set  $A$  is strongly jump traceable, and  $Y \leq_{wtt} \emptyset'$  is ML-random, then  $A \leq_T Y$ . (Greenberg, Nies).

## Definition

A **cost function** is a computable function

$$c : \mathbb{N} \times \mathbb{N} \rightarrow \{x \in \mathbb{Q} : x \geq 0\}.$$

When building a computable approximation of a  $\Delta_2^0$  set  $A$ , we view  $c(x, s)$  as the cost of changing  $A(x)$  at stage  $s$ .

# Obeying a cost function

Recall that  $A$  is  $\Delta_2^0$  iff  $A \leq_T \emptyset'$  iff  $A(x) = \lim_s A_s(x)$  for a computable approximation  $(A_s)_{s \in \mathbb{N}}$  (Limit Lemma).

## Definition

The computable approximation  $(A_s)_{s \in \mathbb{N}}$  **obeys** a cost function  $c$  if

$$\infty > \sum_{x,s} c(x, s) \llbracket x < s \text{ \& } x \text{ is least s.t. } A_{s-1}(x) \neq A_s(x) \rrbracket.$$

We write  $A \models c$  ( $A$  obeys  $c$ ) if **some** computable approximation of  $A$  obeys  $c$ .

Usually we use this to construct some auxiliary object of finite “weight”, such as a bounded request set (aka Kraft-Chaitin set), or a Solovay test.

# Basic existence theorem

For a cost function  $c : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Q}$ , let  $c_{\text{sup}}(x) = \sup_s c(x, s)$ .  
We say that  $c$  has the **limit condition** if  $\lim_x c_{\text{sup}}(x) = 0$ .

## Theorem (Various authors)

*If a cost function  $c$  satisfies the limit condition, then some (promptly) simple set  $A$  obeys  $c$ .*

In fact, one can achieve that  $\forall B \leq_T A [B \models c]$ .

# Implication of cost functions

Cost function  $c(x, s)$  is **monotonic** if it is nonincreasing in  $x$ , and nondecreasing in  $s$ .

For cost functions  $c, d$  we write  $c \longrightarrow d$  if

$$A \models c \text{ implies } A \models d \text{ for each } \Delta_2^0 \text{ set } A.$$

This is equivalent to  $d_{\text{sup}} = O(c_{\text{sup}})$ :

## Theorem (Nies 2009)

*Let  $c, d$  be monotonic cost functions. Suppose  $c$  satisfies the limit condition. Then*

$$c \longrightarrow d \Leftrightarrow \exists N \forall x [Nc_{\text{sup}}(x) > d_{\text{sup}}(x)].$$

*In particular, whether  $A \models c$  only depends on the function  $c_{\text{sup}}$ .*

# Cost function characterization of the $K$ -trivials

The **standard cost function**  $c_K$  is given by

$$c_K(x, s) = \sum_{i=x+1}^s 2^{-K_s(i)}.$$

## Theorem (Nies 05)

$A$  is  $K$ -trivial  $\Leftrightarrow$   
*some computable approximation of  $A$  obeys  $c_K$ .*

## Corollary

*For each  $K$ -trivial  $A$  there is a **c.e.**  $K$ -trivial set  $D \geq_T A$ .*

$D$  is the **change set**  $\{\langle x, i \rangle : A(x) \text{ changes at least } i \text{ times}\}$ .

One verifies that  $D$  obeys  $c_K$  as well.

Actually, this works for any cost function in place of  $c_K$ .

# Proving Kučera's Theorem with a cost function

## Theorem (Kučera 1986)

*Suppose  $Y$  is a ML-random  $\Delta_2^0$  set. Then some promptly simple set  $A$  is Turing below  $Y$ .*

The proof can be phrased in the language of cost functions. Let  $c_Y(x, s) = 2^{-x}$  for each  $x \geq s$ . If  $x < s$ , and  $e < x$  is least such that  $Y_{s-1}(e) \neq Y_s(e)$ , let

$$c_Y(x, s) = \max(c_Y(x, s-1), 2^{-e}).$$

Since  $Y$  is  $\Delta_2^0$ , the cost function  $c_Y$  satisfies the limit condition.

## Proposition (Greenberg and Nies)

*If the  $\Delta_2^0$  set  $A$  obeys  $c_Y$ , then  $A \leq_T Y$  with use function bounded by the identity.*

Some promptly simple  $A$  obeys  $c_Y$ . So  $A \leq_T Y$ .

# Strongly jump traceable sets

- A **computably enumerable trace with bound  $h$**  is a uniformly computably enumerable sequence  $(T_x)_{x \in \mathbb{N}}$  such that  $|T_x| \leq h(x)$  for each  $x$ .
- Let  $J^A(e)$  be the value at  $e$  of a universal  $A$ -partial computable function. (For instance, let  $J^A(e) \simeq \Phi_e^A(e)$  where  $\Phi_e$  is the  $e$ -th Turing functional.)
- The set  $A$  is called **strongly jump traceable** if for **each** order function  $h$ , there is a c.e. trace  $(T_x)_{x \in \mathbb{N}}$  with bound  $h$  such that, whenever  $J^A(x)$  is defined, we have

$$J^A(x) \in T_x$$

- $SJT_{c.e.}$  will denote the class of **c.e.** strongly jump traceable sets.
- There is an incomputable set in  $SJT_{c.e.}$  by Figueira, Nies, Stephan (2004).

# Open questions on cost functions

Usually we are given a class  $\mathcal{D}$  and a cost function  $c$  such that  $A \models c \Rightarrow A \in \mathcal{D}$ . The question is **what else** is in  $\mathcal{D}$ .

## Question

Let  $Y$  be a ML-random  $\Delta_2^0$  set.

- Is there a c.e.  $D \leq_T Y$  such that  $D \not\leq_{\text{wtt}} Y$  (and hence  $D \not\models c_Y$ )?
- If  $B \leq_T Y$  is c.e., is there a c.e.  $A \models c_Y$  such that  $B \leq_T A$ ?

## Question

Let  $Y$  be a Demuth random  $\Delta_2^0$  set. If  $A$  is c.e. and  $A \leq_T Y$ , is  $A$  strongly jump traceable?

I can prove the weaker result that  $A \models c_Y \Rightarrow A$  is s.j.t.

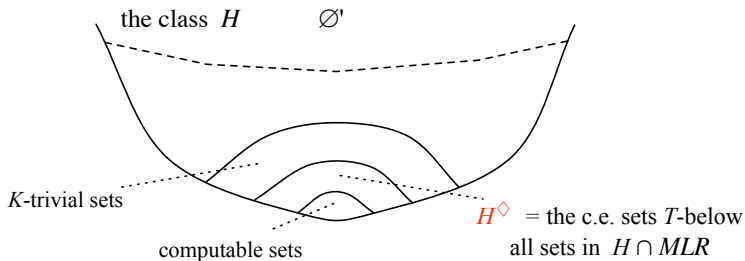
## Part 3

# Classes related to the $K$ -trivials

# Diamond Classes

$2^{\mathbb{N}}$  denotes Cantor space with the uniform (coin-flip) measure.  
For a null class  $\mathcal{H} \subseteq 2^{\mathbb{N}}$ , we define

$\mathcal{H}^{\diamond}$  = the c.e. sets Turing below each ML-random set in  $\mathcal{H}$ .



- The larger  $\mathcal{H}$  is, the smaller is  $\mathcal{H}^{\diamond}$ .
- $\mathcal{H}^{\diamond}$  induces an ideal in the computably enumerable Turing degrees.

# A lowness property and its dual highness property

- Recall that  $Z \subseteq \mathbb{N}$  is **low** if  $Z' \leq_T \emptyset'$ , and  $Z$  is **high** if  $\emptyset'' \leq_T Z'$ .
- These classes are “too big”: we have

$$(\text{low})^\diamond = (\text{high})^\diamond = \text{computable}.$$

(For instance,  $(\text{high})^\diamond = \text{computable}$  because there is a minimal pair of high ML-random sets.)

- So we will try somewhat smaller classes, replacing  $\leq_T$  by the stronger truth-table reducibility  $\leq_{tt}$ .

## Definition (Mohrherr 1986)

A set  $Z$  is **superlow** if  $Z' \leq_{tt} \emptyset'$ .  $Z$  is **superhigh** if  $\emptyset'' \leq_{tt} Z'$ .

A random set can be superlow (low basis theorem). It can also be superhigh but Turing incomplete (Kučera coding).

# These diamond classes coincide with $SJT_{c.e.}$

The following theorems say that a c.e. set  $A$  is strongly jump traceable iff it is computed by many ML-random oracles.

Theorem (Greenberg, Hirschfeldt and Nies (to appear))

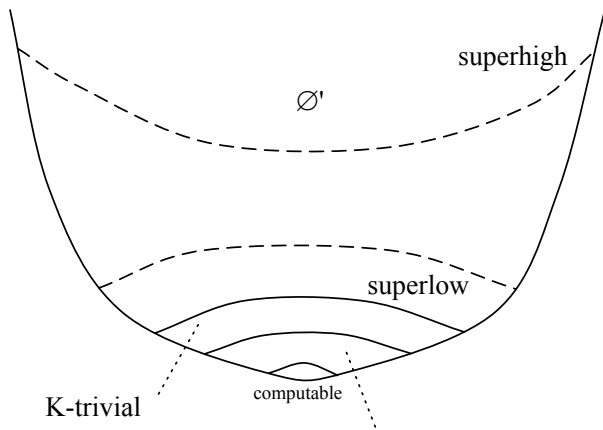
$$SJT_{c.e.} = \text{superlow}^\diamond.$$

*That is, a c.e. set  $A$  is strongly jump traceable  $\Leftrightarrow A$  is Turing below each superlow ML-random set.*

Theorem (Nies (ICALP), improved in G'berg, H'feldt, N)

$$SJT_{c.e.} = \text{superhigh}^\diamond.$$

# Diagram: $SJT_{c.e.}$ means computed by many oracles



$$SJT_{c.e.} = (\text{superlow})^{\diamond} = (\text{superhigh})^{\diamond}$$

# SJTs preserve superlowness

A promising direction is the following application of randomness in degree theory. As a corollary of  $SJT_{c.e.} \subseteq (\text{superlow})^\diamond$ , we have

Theorem (Greenberg and Nies (2008))

Suppose the c.e. set  $A$  is strongly jump traceable. Then

(\*)  $\forall X$  superlow  $[X \oplus A \text{ is superlow}]$ .

This gives a new proof of Diamondstone's result that some promptly simple set is not superlow cuppable.

Question

Is (\*) a characterization of  $SJT_{c.e.}$ ?

Is the class (\*) at least contained in the  $K$ -trivials?

If we restrict (\*) to c.e. sets  $X$ , then it properly contains  $SJT_{c.e.}$  (Diamondstone and Ng, to appear).

# Open questions on classes between $SJT_{c.e.}$ and $K$ -trivial

No natural classes are currently known to lie properly between  $SJT_{c.e.}$  and  $K$ -trivial

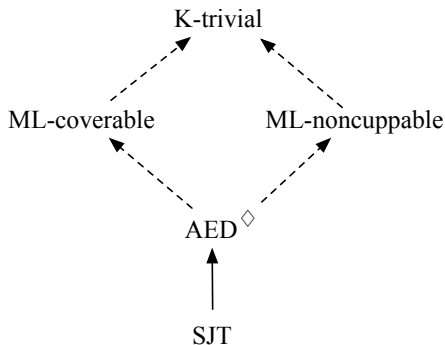
- A good candidate is  $(AED)^\diamond$ .
- Here AED is the class of almost everywhere dominating sets  $D$  of Dobrinen and Simpson: for almost all sets  $X$ , each function  $f \leq_T X$  is dominated by a function  $g \leq_T D$ .
- For the highness properties, there are proper implications

Turing-complete  $\Rightarrow$  AED  $\Rightarrow$  superhigh.

# $(\text{AED})^\diamond$ properly contains $SJT_{c.e.}$ .

- For the corresponding diamond classes, Greenberg and Nies proved that  $SJT_{c.e.}$  is properly contained in  $(\text{AED})^\diamond$ .
- They built a single benign cost function  $c$  such that  $A \models c$  implies  $A \in (\text{AED})^\diamond$ .
- However,  $(\text{AED})^\diamond$  may coincide with  $K$ -trivial.
- This would imply that the classes **ML-coverable** and **ML-noncuppable** also coincide with  $K$ -trivial.

# Classes of c.e. sets between $SJT_{c.e.}$ and $K$ -trivial



(The dashed arrows may be coincidences.)

- $A$  is ML-coverable if  $A \leq_T Y$  for some ML-random  $Y \not\leq_T \emptyset'$ .
- $A$  is ML-noncuppable if  $\emptyset' \leq_T A \oplus Y$  for ML-random  $Y$  implies  $\emptyset' \leq_T Y$ .

Work in progress with Diamondstone and Hirschfeldt shows:  
The class

$$(\omega^\omega\text{-c.e.})^\diamond$$

is a nontrivial proper subclass of  $SJT_{c.e.}$ .

## Some references for Parts 2 and 3

- Downey and Greenberg. They show that **each strongly jump traceable set is  $K$ -trivial**. To appear.
- N. Greenberg and A. Nies. **Benign cost functions and lowness properties**. Submitted.
- N. Greenberg, D. Hirschfeldt and A. Nies. **Characterizing the strongly jump-traceable sets via randomness**. To appear.
- A. Nies. **Calculus of cost functions**. To appear.
- A. Nies. **Computability and randomness**, Oxford, 2009. Sections 5.3, 8.4, 8.5.

## Part 4

# Randomness via computational complexity theory

# Complexity theory: $k$ -traces

## Definition

Let  $k \geq 1$ . A  $k$ -trace is a sequence  $(T_x)_{x \in \Sigma^*}$  of subsets of  $\Sigma^*$  such that

- $|T_x| = k$  for each  $x$
- The function  $x \rightarrow$  (code for)  $T_x$  is in  $P$ .

$(T_x)_{n \in \mathbb{N}}$  is a trace for the function  $f: \Sigma^* \rightarrow \Sigma^*$  if  $f(x) \in T_x$  for each  $x$ .

We say that  $A$  is  $k$ -traceable if each function  $f \in P^A$  has a  $k$ -trace.

In recursion theory, the corresponding notion of  $k$ -traceability (using c.e. traces, tracing functions  $f \leq_T A$ ) implies that  $A$  is  $C$ -trivial, and hence computable.

# Supersparse sets are 2-traceable

## Definition (Ambos-Spies 1986)

Let  $f : \mathbb{N} \mapsto \mathbb{N}$  be a strictly increasing, time constructible function.  $A$  is  **$f$ -super sparse** if

- $A \subseteq \{0^{f(i)} : i \in \mathbb{N}\}$
- Some machine determines  $A(0^{f(i-1)})$  in time  $O(f(i))$ .

Let  $f$  be the iteration of the function  $n \rightarrow 2^n$ . Ambos-Spies constructed an  $f$ -supersparse set in EXPTIME – P.

## Theorem (Ambos-Spies 1986)

*Each  $f$ -supersparse set is 2-traceable.*

## Question

*Is each  $k$ -traceable set low for polynomial randomness [polynomial Schnorr randomness]?*