# Randomness: A Dynamical Point of View

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(Joint work with P. Gács and M. Hoyrup)

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# Randomness

Ingredients:

1. a physical process
2. its observation (measurements)

**Randomness** $\sim$ unpredictability

Mathematical Models:

1. Probability Theory
2. Algorithmic Randomness (Probability + Computability)
3. Ergodic Theory (Dynamical Systems + Probability)

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- Dynamical Randomness (Ergodic Theory + Computability):
  - To follow the typical statistical behavior of all the "effective" processes.
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  1. Introduction to Computable Probability Spaces
     (some new concepts: almost decidable sets, almost computable functions, isomorphisms)
  2. Some structural results about these spaces,
  3. Within this setting, comparison between algorithmic randomness and the dynamical notion,
     (Characterization of Schnorr randomness in terms of its dynamical properties)

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## Algorithmic randomness

### Definitions

## Computable Probability Spaces

### Computability

### Algorithmic Probability Theory

## Dynamical Randomness

### Typicalness

### Typicalness vs Algorithmic Randomness

### Proof Sketch

# Algorithmic randomness

## Martin-Löf

### Definition (Martin-Löf, 1966)

A sequence of sets $(U_n)_{n\in\mathbb{N}}$ is a $\mu$-**Martin-Löf test** if:

- $U_n$ is a constructive open set (uniformly)
- $\mu(U_n)$ decays to zero at a computable rate.

any subset of $\cap_n U_n$ is called a $\mu$-**effective null set**.

### Definition (Martin-Löf, 1966)

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Some generalizations of random objects

1. Random closed sets (Cenzer),
2. Random brownian motion: $(\mathcal{C}_0([0,1]), W)$, with $W$ the Wiener measure (Fouché),
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# Computable metric spaces

We work on:

- computable metric spaces
  (A metric space $(X, d)$ + countable dense subset $S$ s.t. $d|_S$ is computable).

## Examples

1. Euclidean spaces $\mathbb{R}^n$,
2. Cantor space $\{0, 1\}^{\mathbb{N}}$,
3. Continuous functions $\mathcal{C}([0, 1])$,
4. Compact sets $\mathcal{K}(\mathbb{R}^n)$,
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# Computable metric spaces

## An important example

Let $(X, d)$ be a metric space. Let $\mathcal{M}(X)$ be the set of probability measures on $X$.

1. $\mathcal{M}(X)$ has a "natural" topology, the **weak topology**:

$$\mu_n \to \mu \text{ iff } \int f \, \mathrm{d}\mu_n \to \int f \, \mathrm{d}\mu \text{ for all } f \in C_b(X).$$

2. $\mathcal{M}(X)$ has a metric, the **Prokhorov metric** $\pi$, which induces the weak topology.
3. if $(X, d, S)$ is a computable metric space, then $(\mathcal{M}(X), \pi, \mathcal{N})$ is a computable metric space.

$\mathcal{N} = \{\nu_0, \nu_1, \ldots\}$: measures concentrated on finite subsets of $S$.

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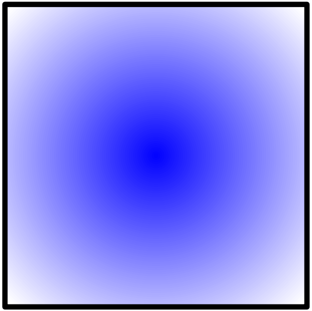
A probability measure $\mu$ is **computable** if it is a computable point of $(\mathcal{M}(X), \pi, \mathcal{N})$.

i.e. there is a computable sequence $\nu_n \in \mathcal{N}$ satisfying $\pi(\nu_n, \mu) < 2^{-n}$.



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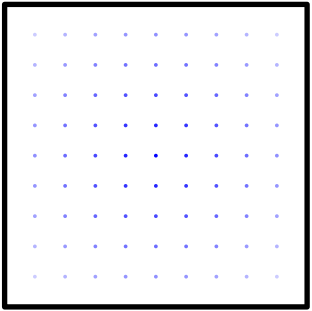

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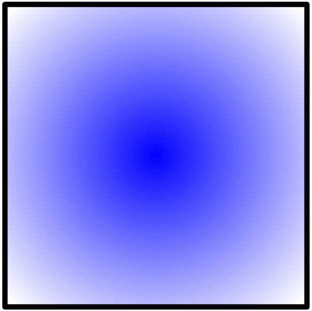

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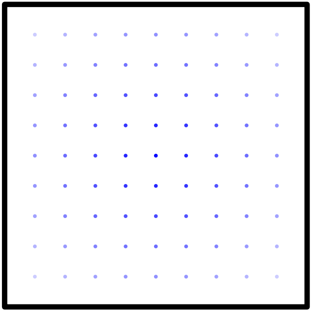

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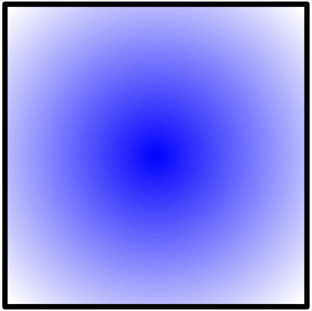

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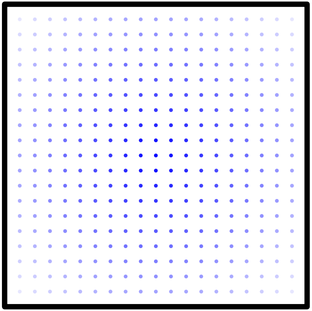

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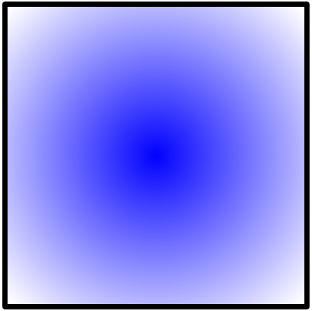

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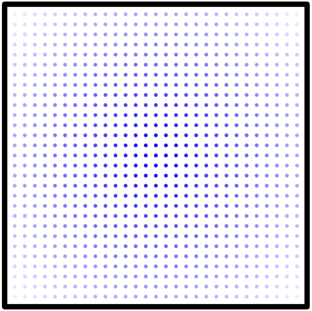

An approximation $\nu_8$ of $\mu$

# Computable probability measure

Let $A \subset X$ be open. In general, the function $\mu \to \mu(A)$ is not computable.

**Example**

For $X = \mathbb{R}$ and $A = (0, 1)$ we have:

- $\delta_{\frac{1}{n}} \to \delta_0$ (weakly), and
- $\delta_{\frac{1}{n}}(A) = 1$ for all $n$ but $\delta_0(A) = 0$.

So $\mu \to \mu(A)$ is not computable since it is even not continuous.

**Theorem (Hoyrup and R.)**

*Let $\mu \in \mathcal{M}(X)$. The following are equivalent:*

- *$\mu$ is computable*
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- $f : X \to Y$ is **almost computable** if it is computable on a constructive $G_\delta$-set of measure one.
- A set $A \subset X$ is **almost decidable** if its indicator function $1_A$ is almost computable.
- An **effective morphism** $f : (X, \mu) \to (Y, \nu)$ is an almost computable function $f : X \to Y$ which maps $\mu$ to $\nu$.

Spaces which are **effectively isomorphic** are then naturally defined.

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Let $(X, \mu)$ be any computable probability space.

Theorem (Hoyrup and R.)

1. *Almost decidable sets form a generating constructive algebra:*
   - *unions and intersections are computable operations,*
   - *almost decidable sets have computable measure.*
2. *$(X, \mu)$ is effectively isomorphic to $(\{0,1\}^{\mathbb{N}}, \nu)$ for some computable $\nu$.*
3. *When $\mu$ has no mass point, $\nu$ can be taken to be the uniform measure*

Application

Transfer of algorithmic randomness.

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# Algorithmic probability theory and Random points

**Theorem**

*When restricting to random points,*

| | $(X, \mu)$ | $\rightsquigarrow$ | $(R_\mu, \mu)$ |
|---|---|---|---|
| *function* | *almost computable* | | *computable* |
| *set* | *almost decidable* | | *decidable* |
| *morphism* | *isomorphism* | | *homeomorphism* |

and so on...

(In particular Morphisms preserve algorithmic randomness.)

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- if $x \in X$ is the state of the system at time $0$, then $T^n(x)$ is the state at time $n$.

# Dynamical Systems

A dynamical system is a pair $(X, T)$ where:

- $X$ is a metric space (the "phase space")
- $T$ is a function from $X$ to itself (the "dynamics")

The idea is that:

- if $x \in X$ is the state of the system at time $0$, then $T^n(x)$ is the state at time $n$.
- Thus, $\{x, T(x), T^2(x), \ldots\}$ represents the trajectory of the system.

# Ergodic theory

The study of dynamical systems from a probabilistic point of view

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- **AND on outcomes** (observations performed over time):
  3. Probabilistic laws shared by sequences of measurements?
     - Ergodic theorems,

# Typical statistical behavior

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"To follow the typical statistical behavior of the system" means to satisfy:

**Theorem (Birkhoff ergodic theorem)**

*Let $f \in L^2$, then with probability one:*

$$A_n^f(x) := \frac{f(x) + f(T(x)) + \ldots + f(T^{n-1}(x))}{n} \longrightarrow \int_X f \, \mathrm{d}\mu$$

# Typical points

## Definition (A classical notion)

$x$ is called **T-typical** if it satisfies the Ergodic theorem for any continuous bounded function $f$.

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## Remark

If we replace continuous by computable, the above definition remains unchanged.

In the computable context, one can suppress the dependency on $T$.

## Definition (typical points)

$x$ is called **typical** if it is $T$-typical for all the effective endomorphisms $T$ that are ergodic.

# Effectively ergodic systems

Let us put

$$P_n^\delta(f) = \mu\left\{x \in X : \sup_{N \geq n} |A_N^f(x) - \int f \, \mathrm{d}\mu| > \delta\right\}.$$

Then the ergodic theorem is equivalent to the following assertion:

$$\lim_{n\to\infty} P_n^\delta(f) = 0 \quad \text{for all } \delta > 0.$$

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$T$ is **effectively ergodic** if we have a computable rate here.

Definition (weakly typical points)

$x$ is called **weakly-typical** if it is $T$-typical for all the effective endomorphisms $T$ that are effectively ergodic.

# Effectively ergodic systems

V'yugin exhibited a (non ergodic) system having uncomputable rate of convergence in Birkhoff theorem.

Problem

do ergodic but not effectively ergodic systems exist?

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do ergodic but not effectively ergodic systems exist?

For any pair $(f, g)$ of integrable functions we put

$$c_n(f,g) := \left| \int f \circ T^n \cdot g \, \mathrm{d}\mu - \int f \, \mathrm{d}\mu \cdot \int g \, \mathrm{d}\mu \right| .$$

$T$ is **mixing** if the correlation coefficients $c_n$ converge to zero.

# Effectively ergodic systems

## and decay of correlations

**Theorem (Hoyrup and R.)**

*Let $s > 1$. If $T$ is $\ln^{-s}$-mixing then it is effectively ergodic.*

# Effectively ergodic systems

## and decay of correlations

### Theorem (Hoyrup and R.)

*Let $s > 1$. If $T$ is $\ln^{-s}$-mixing then it is effectively ergodic.*

### Example

- Independent systems
- Markov systemes
- Hyperbolic systems
- Piecewise-expanding systems
- ...etc

# Typicalness vs Algorithmic Randomness

Let $(X, \mu)$ be a computable probability space.

Theorem (Corollary of (V'yugin, 1998))

*Martin-Löf random* $\Longrightarrow$ *typical.*

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*Schnorr random* $\Longleftrightarrow$ *weakly-typical*

# Typicalness vs Algorithmic Randomness

## Theorem

| | |
|---|---|
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| (⇑) | (⇑) |
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# Typicalness vs Algorithmic Randomness

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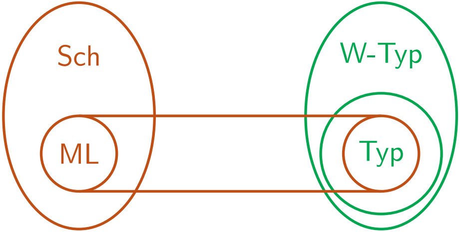

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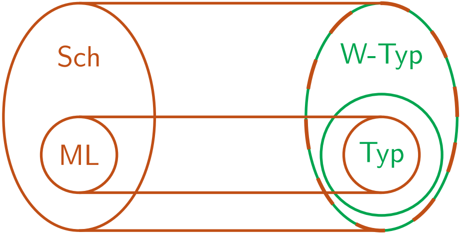

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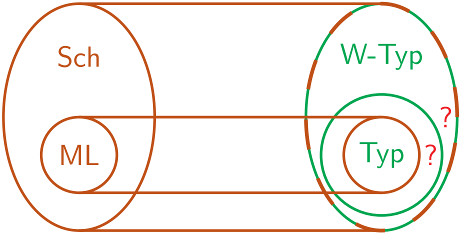

# Sketch of the proof

## Schnorr random ⇒ Weakly typical:

Let $T$ be effectively ergodic and $f$ almost computable. Put

$$D_n^\delta = \left\{ x \in X : \sup_{N \geq n} \left| A_N^f(x) - \int f \, \mathrm{d}\mu \right| > \delta \right\},$$

Show that $\exists\ \delta_k \to 0$ such that, for each $k$, no Schnorr random point belongs to

$$\bigcap_n D_n^{\delta_k}.$$

# Sketch of the proof

## Schnorr random ⇒ Weakly typical:

Since $T$ and $f$ are almost computable, so are the functions:

$$\overline{f}_N := \left| A_N^f(x) - \int f \, \mathrm{d}\mu \right|,$$

then we prove the following:

### Lemma

*If functions $f_i : X \to \mathbb{R}$ are uniformly almost computable, then there exists a computable sequence $(x_k)$ satisfying:*

- $x_k$ *is dense,*
- $\mu\left(\{f_i^{-1}(x_k)\}\right) = 0.$

# Sketch of the proof

Schnorr random ⇒ Weakly typical:

Hence, there exists a computable sequence $\delta_k \to 0$ such that:

$$\overline{f}_N^{-1}(\delta_k, \infty) = \left\{ x \in X : \left| A_N^f(x) - \int f \, \mathrm{d}\mu \right| > \delta_k \right\}$$

are uniformly almost decidable sets.

The construction of the required Schnorr test follows.

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Weakly typical ⇒ Schnorr random:

If $x$ is not Schnorr random, we construct an effectively ergodic system $T$ such that $x$ is not $T$-typical.

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Weakly typical ⇒ Schnorr random:

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We use the following modification of a result by Schnorr:

## Proposition

*If the sequence $\omega \in (\{0,1\}^{\mathbb{N}}$, uniform measure) is not Sch-random, then there exists an isomorphism $\varphi$ of $(\{0,1\}^{\mathbb{N}}$, uniform measure) such that $\varphi(\omega)$ does not satisfy the law of large numbers.*

That is, the sequence $\varphi(\omega)$ is not typical for the shift transformation $\sigma$, defined by:

$$\sigma(\omega_0\omega_1\omega_2...) = \omega_1\omega_2\omega_3...$$

# Sketch of the proof

Weakly typical $\Rightarrow$ Schnorr random:

We use the isomorphism Theorem to transfer the dynamic on Cantor space to the general computable probability space:

If $\phi$ is the isomorphism between $(X, \mu)$ and $(\{0,1\}^{\mathbb{N}}, \text{uniform measure})$ and $\psi = \phi \circ \varphi$, then

$$T := \psi^{-1} \circ \sigma \circ \psi$$

satisfy the required properties.

Remark

$T$ has exponential decay of correlations. Hence $T$ is effectively ergodic.

Este es el fin. Gracias.