On Domains of Universal Machines

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joint work with *Cristian S. Calude, André Nies* and *Frank Stephan*

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Main References

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Outline

1. Prefix Codes
2. Description complexity
3. Universal codes
4. Density
5. Spectral properties
6. Plain machines

Prefix Codes

Fix an alphabet $X = \{0, \ldots, r-1\}$, $r \geq 2$, and denote by X^* the set of finite strings (words) on X.

Definition (Prefix Code)

*A subset V of X^* is called a **prefix code** provided $w \sqsubseteq v$ and $w, v \in V$ imply $w = v$.*

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A subset V of X^* is called a **prefix code** *provided $w \sqsubseteq v$ and $w, v \in V$ imply $w = v$.*

Proposition (Kraft-McMillan inequality)

Every (prefix) code $V \subseteq X^*$ satisfies $\sum_{w \in V} r^{-|w|} \leq 1$.

Prefix Maximality

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$V \subseteq X^*$ is a **prefix maximal** *code provided V is a prefix code and for every prefix code $W \subseteq X^*$, $V \subseteq W$ implies $W = V$.*

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Fact (Sufficient condition)

A prefix code V is prefix maximal if $\sum_{w \in V} r^{-|w|} = 1$.

KRAFT's construction

Theorem (KRAFT's construction)

Let $s : \mathbb{N} \to \mathbb{N}$ *be a function such that* $s(0) = 0$ *and* $\sum_{n\in\mathbb{N}} s(n) \cdot r^{-n} \leq 1$*. Then there is a prefix code* $V \subseteq X^*$ *such that*

$$\big|\{w : w \in V \land |w| = n\}\big| = s(n)\,.$$

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Observe that $\sum_{n\in\mathbb{N}} s(n) = |V|$.

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Theorem (Maximality of infinite prefix codes)

Let $s : \mathbb{N} \to \mathbb{N}$ *be a function such that* $\sum_{n\in\mathbb{N}} s(n) \cdot r^{-n} \leq 1$ *and* $\sum_{n\in\mathbb{N}} s(n) = \infty$. *Then there is a prefix maximal code* $V \subseteq X^*$ *such that*

$$\big|\{w : w \in V \wedge |w| = n\}\big| = s(n)\,.$$

Computability

Theorem (Kraft-Chaitin)

Let $s : \mathbb{N} \to \mathbb{N}$ be a left computable (approximable from below) function such that $s(0) = 0$ and $\sum_{n \in \mathbb{N}} s(n) \cdot r^{-n} \leq 1$. Then there is a computably enumerable prefix code $V \subseteq X^*$ such that

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$$\left|\{w : w \in V \wedge |w| = n\}\right| = s(n)\,.$$

Corollary

If, moreover, $s : \mathbb{N} \to \mathbb{N}$ is a computable function then V is also computable.

Description complexity: plain complexity

Definition (Description complexity K_φ)

Let $\varphi :\subseteq X^* \to X^*$ be a partial computable function.

$$K_\varphi(w) := \inf\{|\pi| : \varphi(\pi) = w\}$$

Definition (Plain or Simple universal machine)

A machine (mapping) $\mathfrak{U}_S :\subseteq X^* \to X^*$ is called **universal** *if and only if for every partial computable mapping $\varphi :\subseteq X^* \to X^*$ there is a constant c_φ such that*

$$\forall w (K_\varphi(w) \leq K_{\mathfrak{U}_S}(w) + c_\varphi) .$$

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Definition (Plain description complexity)

$$C(w) := \min\{|\pi| : \mathfrak{U}_S(\pi) = w\}$$

Description complexity: prefix-free complexity

Definition (Prefix-free universal machine)

A prefix-free machine (mapping) $\mathfrak{U}_P :\subseteq X^* \to X^*$ *is called* **universal** *if and only if*

1. $\mathrm{dom}(\mathfrak{U}_P)$ *is prefix-free, and*
2. *for every partial computable mapping* $\varphi :\subseteq X^* \to X^*$ *with prefix-free domain* $\mathrm{dom}(\varphi)$ *there is a constant* c_φ *such that*

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Definition (Prefix-free description complexity)

$$H(w) := \min\{|\pi| : \mathfrak{U}_P(\pi) = w\}$$

Numbering words

Definition (Quasi-lexicographical order of $\{0, \ldots, r-1\}^*$)

$$w <_{ql} v :\Longleftrightarrow |w| < |v| \vee \left(|w| = |v| \to 0.w <_{real} 0.v\right)$$

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Not to confuse with the lexicographical order:

$$e <_{\text{lex}} 0 <_{\text{lex}} 00 <_{\text{lex}} \cdots <_{\text{lex}} 0^i <_{\text{lex}} \dots$$

$$\dots \quad <_{\text{lex}} 0^\ell 1 <_{\text{lex}} 0^\ell 10 <_{\text{lex}} \cdots <_{\text{lex}} 01 <_{\text{lex}} \cdots <_{\text{lex}} 1 <_{\text{lex}} \dots$$

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Definition (Complexity of natural numbers)

$$K_\varphi(n) := K_\varphi(n\text{-th word in } X^* \text{ w.r.t. } \leq_{ql})$$

Universal c.e. prefix codes

Definition

We say that a computably enumerable prefix code $V \subseteq X^*$ is ***universal*** *if there is a universal prefix-free machine $\mathfrak{U}$ such that $V \supseteq \mathrm{dom}(\mathfrak{U})$.*

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- Characterise universal c.e. prefix codes.

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- Characterise universal c.e. prefix codes.
- Are universal c.e. prefix codes computable?
- How big (set-theoretic, information-theoretic) are universal c.e. prefix codes?
- Are universal c.e. prefix codes necessarily domains of prefix-free universal machines?

A characterisation of universal c.e. prefix codes 1

Theorem

Let $V \subseteq X^*$ be a c.e. prefix code. Then, the following statements are equivalent:

1. *The set V is a universal c.e. prefix code.*
2. *For every c.e. prefix code $D \subseteq X^*$ there exist a partial computable one-one function $\varphi :\subseteq X^* \to X^*$ and a constant $k \in \mathbb{N}$ such that:*
   a. $D \subseteq \mathrm{dom}(\varphi)$, $\varphi(D) \subseteq V$, *and*
   b. $|\varphi(u)| \leq |u| + k$, *for every* $u \in \mathrm{dom}(\varphi)$.

A characterisation of universal c.e. prefix codes 2

For the case $V = \mathrm{dom}(\mathfrak{U})$, where $\mathfrak{U}$ is a prefix-free universal machine we have:

Corollary

For every c.e. prefix code $D \subseteq X^*$ and every universal prefix machine $\mathfrak{U}$ there are a one-one partial computable function $\varphi :\subseteq X^* \to X^*$ and a constant $k \in \mathbb{N}$ such that:

a. $D \subseteq \mathrm{dom}(\varphi)$, $\varphi(D) \subseteq \mathrm{dom}(\mathfrak{U})$,

b. $|\varphi(u)| \le |u| + k$, *for all* $u \in D$, *and*

c. $\mathfrak{U}(\varphi(u)) = u$, *for all* $u \in D$.

Computability and set-theoretical maximality

Theorem (Nies, Calude & *St.*)

No universal c.e. prefix code $V \subseteq X^*$ *is computable.*

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No universal c.e. prefix code $V \subseteq X^*$ *is computable.*

Lemma

If $V \subseteq X^*$ *is a c.e. prefix maximal code, then* V *is computable.*

Computability and set-theoretical maximality

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No universal c.e. prefix code $V \subseteq X^*$ is computable.

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If $V \subseteq X^*$ is a c.e. prefix maximal code, then V is computable.

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No universal c.e. prefix code is a prefix maximal code.

Computability and set-theoretical maximality

Theorem (Nies, Calude & *St.*)

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Lemma

If $V \subseteq X^*$ is a c.e. prefix maximal code, then V is computable.

Corollary

No universal c.e. prefix code is a prefix maximal code.

However:
There are computable prefix codes which are *not* contained in a computable prefix maximal code.

Spectrum Function

Definition (Spectrum Function)

Let $L \subseteq X^*$. Then

$$s_L(n,c) := \left|\{w : w \in L \wedge n \leq |w| \leq n+c\}\right|$$

is referred to as the ***spectrum function*** *of L.*

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Specials cases

- $s_L(n) := s_L(n,0)$ is also referred to as the *structure function* of the language $L \subseteq X^*$ [Chomsky and Miller '58].
- $C_L(n) := s_L(0,n)$ is also referred to as the *census function* of the language $L \subseteq X^*$.

Spectrum Function 2

Fact

If $W \subseteq X^*$ is computably enumerable (computable) then s_W is a left computable (computable) function, and if W is computably enumerable and s_W is a computable function then W is also computable.

Spectrum Function

Fact

If $W \subseteq X^*$ is computably enumerable (computable) then s_W is a left computable (computable) function, and if W is computably enumerable and s_W is a computable function then W is also computable.

Fact (Uniform embedding)

If $W, W' \subseteq X^*$ are computably enumerable and if there is a $c \in \mathbb{N}$ such that $s_W(n,c) \leq s_{W'}(n,c)$ for all $n \in \mathbb{N}$ then there is a one-to-one partial computable function $\varphi : W \to W'$ such that $\big||\varphi(w)| - |w|\big| \leq c$ for all $w \in W$.

Entropy and Logarithmic Density

Fact

If $V \subseteq X^*$ is a (prefix) code then

$$\lim_{n \to \infty} r^{-n} \cdot s_V(0,n) = 0.$$

Definition (Entropy)

$$\mathrm{H}_L := \inf\left\{\alpha : \alpha \geq 0 \wedge \sum_{w \in L} r^{-\alpha \cdot |w|} < \infty\right\} = \limsup_{n \to \infty} \frac{\log_r\left(1 + s_L(0,n)\right)}{n}$$

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Definition (Logarithmic Density)

$$\underline{\mathrm{H}}_L := \liminf_{n\to\infty} \frac{\log_r\big(1 + s_L(0,n)\big)}{n}$$

Entropy and Logarithmic Density: Properties

Fact

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\begin{aligned}
0 &\leq \underline{\mathrm{H}}_L \leq \mathrm{H}_L \leq 1, \\
\underline{\mathrm{H}}_L &\leq \underline{\mathrm{H}}_{L'}, \quad \textit{if } L \subseteq L' \textit{, and} \\
\mathrm{H}_{L \cup L'} &= \max\{\mathrm{H}_L, \mathrm{H}_{L'}\}
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Lemma

If $V \subseteq X^*$ is a universal c.e. prefix code then $\sum_{w \in L} r^{-|w|} < 1$, $\sum_{w \in L} r^{-\alpha \cdot |w|} = \infty$ *for* $\alpha < 1$ *and* $\underline{\mathrm{H}}_V = 1$.

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$$\begin{aligned} 0 &\leq \underline{\mathrm{H}}_L \leq \mathrm{H}_L \leq 1, \\ \underline{\mathrm{H}}_L &\leq \underline{\mathrm{H}}_{L'}, \quad \textit{if } L \subseteq L' \textit{, and} \\ \mathrm{H}_{L \cup L'} &= \max\{\mathrm{H}_L, \mathrm{H}_{L'}\} \end{aligned}$$

Lemma

If $V \subseteq X^*$ is a universal c.e. prefix code then $\sum_{w \in L} r^{-|w|} < 1$, $\sum_{w \in L} r^{-\alpha \cdot |w|} = \infty$ for $\alpha < 1$ and $\underline{\mathrm{H}}_V = 1$.

However:

There are also computable prefix codes $V \subseteq X^*$ which satisfy $\sum_{w \in L} r^{-|w|} < 1$, $\sum_{w \in L} r^{-\alpha \cdot |w|} = \infty$ for $\alpha < 1$ and $\underline{\mathrm{H}}_V = 1$.

SOLOVAY's universal prefix machine $\mathfrak{T}$

Proposition

Let $\mathfrak{T}$ be SOLOVAY's universal prefix machine. Then there are an $n_0 \in \mathbb{N}$ and a $d \in \mathbb{N}$ such that

$$r^{n-H(n)-d} \leq s_{\mathrm{dom}(\mathfrak{T})}(n) \leq r^{n-H(n)+d} \text{ for all } n \geq n_0 \,.$$

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Lemma

Let $W \subseteq X^*$ be computably enumerable and $\sum_{w \in W} r^{-|w|} < \infty$. Then there is a $d \in \mathbb{N}$ such that

$$s_W(0,n) \leq r^{n-H(n)+d} \text{ for all } n \in \mathbb{N} \,.$$

Spectral characterisations

Theorem (Universal c.e. prefix codes)

Let $V \subseteq X^*$ be a computably enumerable prefix code. Then V is a universal c.e. prefix code if and only if there are $c, d \in \mathbb{N}$ such that

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Theorem (Domains of universal prefix-free machines)

Let $W \subseteq X^*$ be a computably enumerable prefix code. Then W is the domain of a universal prefix-free machine $\mathfrak{U}$ if and only if there is a constant $c \in \mathbb{N}$ such that

$$H(\langle n, s_W(n,c)\rangle) \geq n \text{ for all } n \in \mathbb{N}.$$

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There is a universal c.e. prefix code which is not the domain of a universal prefix-free machine.

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Construction: Let $\mathfrak{U}$ be a universal prefix-free machine with $\sum_{w\in\mathrm{dom}(U)} r^{-|w|} < r^{-1}$ and let

$$s(n) := \begin{cases} 0\,, & \text{if } s_{\mathrm{dom}(\mathfrak{U})}(n) = 0\text{ , and} \\ r^{\lceil \log_r s_{\mathrm{dom}(\mathfrak{U})}(n)\rceil}\,, & \text{otherwise.} \end{cases}$$

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Then $s : \mathbb{N} \to \mathbb{N}$ is left computable, $s \geq s_{\mathrm{dom}(\mathfrak{U})}$ and $\sum_{n \in \mathbb{N}} s(n) \cdot r^n \leq 1$.

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Then $s : \mathbb{N} \to \mathbb{N}$ is left computable, $s \geq s_{\mathrm{dom}(\mathfrak{U})}$ and $\sum_{n\in\mathbb{N}} s(n)\cdot r^n \leq 1$.

Thus there is a (universal) c.e. prefix code V such that $s_V = s$.

Observe that $s_V(n,c)$ has the form $r^{m_1} + \cdots + r^{m_k}$ with $m_i \leq n+c$ for some $k \leq c+1$. Hence $H(\langle n, s_V(n,c)\rangle) = O(\log n)$.

CHAITIN's Ω-number

Proposition

If $V \subseteq X^*$ is a universal c.e. prefix code then the number

$$\Omega_V = \sum_{w \in V} r^{-|w|}$$

is a left computable MARTIN-LÖF*-random real.*

Domains of plain machines

Theorem (Supersets of domains)

Let $W \subseteq X^*$ *be computably enumerable. Then* $W \supseteq \mathrm{dom}(\mathfrak{U})$ *for a plain universal machine* $\mathfrak{U}$ *if and only if there is a* $c \in \mathbb{N}$ *such that*

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Theorem (Domains of universal plain machines)

A computably enumerable set $W \subseteq X^*$ is a domain of a universal plain machine if and only if there is a constant $c \in \mathbb{N}$ such that

$$C(s_W(n,c)) \ge n \text{ for all } n \in \mathbb{N}.$$