

A statistical mechanical interpretation of algorithmic information theory

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Abstract

In this talk, we develop a statistical mechanical interpretation of Algorithmic Information Theory (AIT, for short).

We introduce the notion of thermodynamic quantities, such as free energy, energy, (statistical mechanical) entropy, and specific heat, into AIT. We then investigate their properties from the point of view of algorithmic randomness. As a result, we see that, in this statistical mechanical interpretation, the temperature equals to the partial randomness (and therefore compression rate) of the values of all these thermodynamic quantities, which include the temperature itself.

Reflecting this self-referential nature of the temperature, we obtain a fixed point theorems on partial randomness.

Preliminaries: Program-size Complexity

- $\{0, 1\}^* := \{\lambda, 0, 1, 00, 01, 10, 11, 000, \dots\}$. The set of finite binary strings.
- For any $s \in \{0, 1\}^*$, $|s|$ denotes the length of s .
- Let $V \subset \{0, 1\}^*$. We say V is a prefix-free set if for any distinct s and $t \in V$, s is not a prefix of t .

For example $\{0, 10\}$: prefix-free $\{0, 01\}$: not prefix-free

U : universal self-delimiting Turing machine.

$\text{Dom } U$, i.e., the domain of definition of U , is a prefix-free set.

Definition The program-size complexity (or Kolmogorov complexity) $H(s)$ of $s \in \{0, 1\}^*$ is defined by

$$H(s) := \min \left\{ |p| \mid p \in \{0, 1\}^* \ \& \ U(p) = s \right\}.$$

$H(s)$: The length of the shortest input for the universal self-delimiting Turing machine U to output s . \Rightarrow $H(s)$: The degree of randomness of s .
(the size of compressed s)

Preliminaries: Randomness of Real Number

Let $\alpha \in \mathbb{R}$. $\alpha \upharpoonright n$ denotes the first n bits of the base-two expansion of $\alpha - \lfloor \alpha \rfloor$.
The fractional part of α .

Definition [weak Chaitin randomness, Chaitin 1975]

We say $\alpha \in \mathbb{R}$ is weakly Chaitin random if $n \leq H(\alpha \upharpoonright n) + O(1)$,
i.e., any prefix of the base-two expansion of α cannot be compressed by H .

This notion is equivalent to Martin-Löf randomness.

Definition [Chaitin's halting probability Ω , Chaitin 1975]

$$\Omega := \sum_{p \in \text{Dom } U} 2^{-|p|}.$$

The first n bits of the base-two expansion of Ω solve the halting problem of U for inputs of length at most n .

Theorem [Chaitin 1975] Ω is weakly Chaitin random.

Preliminaries: Partial Randomness of Real Number

The partial randomness (degree of randomness) of a real can be characterized by a real.

Definition [weak Chaitin D -randomness, Tadaki 2002] Let $D \in [0, 1]$.

We say $\alpha \in \mathbb{R}$ is weakly Chaitin D -random if $Dn \leq H(\alpha \upharpoonright n) + O(1)$ for $\forall n$.

In the case of $D = 1$, the weak Chaitin D -randomness results in the weak Chaitin randomness.

Definition [D -compressibility] Let $D \in [0, 1]$.

We say $\alpha \in \mathbb{R}$ is D -compressible if $H(\alpha \upharpoonright n) \leq Dn + o(n)$,

which is equivalent to $\limsup_{n \rightarrow \infty} \frac{H(\alpha \upharpoonright n)}{n} \leq D$.

Remark If $\alpha \in \mathbb{R}$ is weakly Chaitin D -random and D -compressible, then

$$\lim_{n \rightarrow \infty} \frac{H(\alpha \upharpoonright n)}{n} = D.$$

The compression rate of α by program-size complexity is equal to D .

⟨The converse does not necessarily hold.⟩

Preliminaries: Generalization of Ω

Definition [generalization of Chaitin's Ω , Tadaki 1999]

$$\Omega(D) := \sum_{p \in \text{Dom } U} 2^{-\frac{|p|}{D}} \quad (D > 0).$$

$\Omega(1) = \Omega$. If $0 < D \leq 1$, then $\Omega(D) \leq \Omega$ and $\Omega(D)$ converges, in particular.

Theorem [Tadaki 1999] Let $D \in \mathbb{R}$.

(i) If $0 < D \leq 1$ and D is **computable**, then $\Omega(D)$ is weakly Chaitin D -random and D -compressible.

\Rightarrow The partial randomness (compression rate) of $\Omega(D)$ equals to D .

(ii) If $1 < D$, then $\Omega(D)$ diverges to ∞ . □

Here, we say that a real D is computable if the mapping $\mathbb{N}^+ \ni n \mapsto D \upharpoonright n$ is a total recursive function.

Motivation

[Calude & Stay, Information and Computation 204 (2006)] pointed out that $\Omega(D)$ is similar to a partition function in statistical mechanics.

- In statistical mechanics, the partition function Z is given as:

$$Z = \sum_n e^{-\frac{E_n}{kT}}.$$

Here, n denotes the quantum number of an energy eigenstate of a quantum system, E_n its energy, and T temperature.

- On the other hand, $\Omega(D)$ is given as:

$$\Omega(D) = \sum_{p \in \text{Dom } U} 2^{-\frac{|p|}{D}} \quad (D > 0).$$

Thus, Z coincides with $\Omega(D)$ by performing the following replacements:

An energy eigenstate n	\Rightarrow	A program $p \in \text{Dom } U$,
The energy E_n of n	\Rightarrow	The length $ p $ of p ,
Temperature T	\Rightarrow	Compression rate D ,
Boltzmann constant k	\Rightarrow	$1/\ln 2$.

What is the partition function in statistical mechanics ?

Quick Review of Statistical Mechanics (I)

Consider a quantum system **at constant temperature T** .

(That is, imagine a quantum system in thermal contact with a very large heat reservoir at constant temperature T)

Statistical mechanics states that the probability $\text{Prob}(n)$ that the system is in an energy eigenstate n with energy E_n is given as:

$$\text{Prob}(n) = \frac{1}{Z} e^{-\frac{E_n}{kT}}.$$

Here, the normalization factor $Z := \sum_n e^{-\frac{E_n}{kT}}$ is called the partition function of the quantum system.

The partition function Z is of particular importance in statistical mechanics, because **all the thermodynamic quantities of the quantum system can be expressed by using the partition function Z , and the knowledge of Z is sufficient to understand all the macroscopic properties of the system.**

Quick Review of Statistical Mechanics (II)

Thermodynamic Quantities of the quantum system at temperature T

- **Free Energy** $F = -kT \ln Z$.

The free energy F of a quantum system is related to the work performed by the system during a process at constant temperature T .

- **Energy** $E = \sum_n E_n \text{Prob}(n) = \frac{1}{Z} \sum_n E_n e^{-\frac{E_n}{kT}} = kT^2 \frac{d}{dT} \ln Z$.

Thus, the energy E of the quantum system is the expected value of an energy E_n of an energy eigenstate n .

- **(Statistical Mechanical) Entropy** $S = \frac{E - F}{T}$.

Note that the entropy S of the system equals to the Shannon entropy of the probability distribution $\{\text{Prob}(n)\}$, i.e., $S = -k \sum_n \text{Prob}(n) \ln \text{Prob}(n)$.

- **Specific Heat** $C = \frac{dE}{dT}$.

Aim of this talk

We propose a statistical mechanical interpretation of AIT (Algorithmic Information Theory) where $\Omega(D)$ appears as a partition function.

We do this in the following manner:

We introduce the notion of thermodynamic quantities such as free energy, energy, (statistical mechanical) entropy, and specific heat into AIT by performing the following replacements for the corresponding thermodynamic quantities of a quantum system at temperature T :

- An energy eigenstate n \Rightarrow A program $p \in \text{Dom } U$,
- The energy E_n of n \Rightarrow The length $|p|$ of p ,
- Boltzmann constant k \Rightarrow $1/\ln 2$.

We then determine the convergence or divergence of each of the quantities. In the case where a thermodynamic quantity converges, we calculate the partial randomness (compression rate) of the values of the thermodynamic quantity, based on program-size complexity $H(s)$.

\Rightarrow We see that all of the partial randomness of the thermodynamic quantities, which include temperature T itself, equal to T .

Immediate Application of the Replacements: **Transient Definitions**

Perform the following replacements for the corresponding thermodynamic quantities of a quantum system at temperature T . (U : optimal computer)

An energy eigenstate n \Rightarrow A program $p \in \text{Dom } U$,

The energy E_n of n \Rightarrow The length $|p|$ of p ,

Boltzmann constant k \Rightarrow $1/\ln 2$. **Boltzmann factor:** $2^{-\frac{|p|}{T}}$

Partition function $Z(T) = \sum_n e^{-\frac{E_n}{kT}} \Rightarrow Z(T) = \sum_{p \in \text{Dom } U} 2^{-\frac{|p|}{T}},$

Free energy $F(T) = -kT \ln Z(T) \Rightarrow F(T) = -T \log_2 Z(T),$

Energy $E(T) = \frac{1}{Z(T)} \sum_n E_n e^{-\frac{E_n}{kT}} \Rightarrow E(T) = \frac{1}{Z(T)} \sum_{p \in \text{Dom } U} |p| 2^{-\frac{|p|}{T}},$

Entropy $S(T) = \frac{E(T) - F(T)}{T} \Rightarrow S(T) = \frac{E(T) - F(T)}{T},$

Specific heat $C(T) = \frac{d}{dT} E(T) \Rightarrow C(T) = \frac{d}{dT} E(T).$

Thermodynamic Quantities in AIT: **Rigorous Definitions**

Redefine the transient definitions rigorously as follows.

Definition Let q_1, q_2, q_3, \dots be an arbitrary enumeration of $\text{Dom } U$. \square

Note that the results of this talk are independent of the choice of $\{q_i\}$.

Definition [Thermodynamic Quantities in AIT, Tadaki 2008] Let $T > 0$.

(i) **partition function** $Z(T) := \lim_{m \rightarrow \infty} Z_m(T)$, where $Z_m(T) = \sum_{i=1}^m 2^{-\frac{|q_i|}{T}}$.

(ii) **free energy** $F(T) := \lim_{m \rightarrow \infty} F_m(T)$, where $F_m(T) = -T \log_2 Z_m(T)$.

(ii) **energy** $E(T) := \lim_{m \rightarrow \infty} E_m(T)$, where $E_m(T) = \frac{1}{Z_m(T)} \sum_{i=1}^m |q_i| 2^{-\frac{|q_i|}{T}}$.

(iii) **entropy** $S(T) := \lim_{m \rightarrow \infty} S_m(T)$, where $S_m(T) = \frac{E_m(T) - F_m(T)}{T}$.

(iv) **specific heat** $C(T) := \lim_{m \rightarrow \infty} C_m(T)$, where $C_m(T) = E'_m(T)$. \square

Remark These are **variants of Chaitin's Ω** . In particular, $Z(T) = \Omega(T)$.

Temperature = Partial Randomness.

Statistical Mechanical Quantity in AIT: **Partition Function**

Recall that, in statistical mechanics, the partition function Z is given by:

$$Z = \sum_n e^{-\frac{E_n}{kT}}.$$

Definition [partition function] For each $m \in \mathbb{N}^+$ and each real $T > 0$, $Z_m(T)$ of finite size m is defined as

$$Z_m(T) := \sum_{i=1}^m 2^{-\frac{|q_i|}{T}}.$$

Then, for each $T > 0$, the partition function $Z(T)$ is defined as

$$Z(T) := \lim_{m \rightarrow \infty} Z_m(T).$$

$$Z(T) = \Omega(T).$$

Theorem [partition function, Tadaki 1999, 2002] Let $T \in \mathbb{R}$.

(i) If $0 < T \leq 1$ and T is **computable**, then $Z(T)$ converges to a left-computable real which is **weakly** Chaitin T -random and T -compressible.

\Rightarrow The partial randomness of $Z(T)$ equals to the temperature T .

(ii) If $1 < T$, then $Z(T)$ diverges to ∞ .

\Rightarrow "Phase Transition" occurs at temperature 1.

Thermodynamic Quantities in AIT: **Free Energy**

Recall that, in statistical mechanics, the free energy F is given by:

$$F = -kT \ln Z.$$

Definition [free energy] For each $m \in \mathbb{N}^+$ and each real $T > 0$, $F_m(T)$ of finite size m is defined as

$$F_m(T) := -T \log_2 Z_m(T).$$

Then, for each $T > 0$, the free energy $F(T)$ is defined as

$$F(T) := \lim_{m \rightarrow \infty} F_m(T).$$

Theorem [free energy] Let $T \in \mathbb{R}$.

(i) If $0 < T \leq 1$ and T is **computable**, then $F(T)$ converges to a right-computable real which is **weakly** Chaitin T -random and T -compressible.

\Rightarrow The partial randomness of $F(T)$ equals to the temperature T .

(ii) If $1 < T$, then $F(T)$ diverges to $-\infty$.

\Rightarrow “Phase Transition” occurs at temperature 1.

Thermodynamic Quantities in AIT: **Energy**

Recall that, in statistical mechanics, the energy E is given by:

$$E = \frac{1}{Z} \sum_n E_n e^{-\frac{E_n}{kT}}.$$

Definition [energy] For each $m \in \mathbb{N}^+$ and each real $T > 0$, $E_m(T)$ of finite size m is defined as

$$E_m(T) := \frac{1}{Z_m(T)} \sum_{i=1}^m |q_i| 2^{-\frac{|q_i|}{T}} = \frac{\sum_{i=1}^m |q_i| 2^{-|q_i|/T}}{\sum_{i=1}^m 2^{-|q_i|/T}}.$$

Then, for each $T > 0$, the energy $E(T)$ is defined by $E(T) := \lim_{m \rightarrow \infty} E_m(T)$.

Definition We say $\alpha \in \mathbb{R}$ is Chaitin T -random if $\lim_{n \rightarrow \infty} H(\alpha \upharpoonright n) - Tn = \infty$.

Theorem [energy] Let $T \in \mathbb{R}$.

(i) If $0 < T < 1$ and T is **computable**, then $E(T)$ converges to a left-computable real which is Chaitin T -random and T -compressible.

\Rightarrow The partial randomness of $E(T)$ equals to the temperature T .

(ii) If $1 \leq T$, then $E(T)$ diverges to ∞ .

\Rightarrow “Phase Transition” occurs at temperature 1.

Thermodynamic Quantities in AIT: **Entropy**

Recall that, in statistical mechanics, the entropy S is given by:

$$S = \frac{E - F}{T}.$$

Definition [entropy] For each $m \in \mathbb{N}^+$ and each real $T > 0$, $S_m(T)$ of finite size m is defined as

$$S_m(T) := \frac{E_m(T) - F_m(T)}{T}.$$

Then, for each $T > 0$, the entropy $S(T)$ is defined as

$$S(T) := \lim_{m \rightarrow \infty} S_m(T).$$

Theorem [entropy] Let $T \in \mathbb{R}$.

(i) If $0 < T < 1$ and T is **computable**, then $S(T)$ converges to a left-computable real which is Chaitin T -random and T -compressible.

\Rightarrow The partial randomness of $S(T)$ equals to the temperature T .

(ii) If $1 \leq T$, then $S(T)$ diverges to ∞ .

\Rightarrow “Phase Transition” occurs at temperature 1.

Thermodynamic Quantities in AIT: **Specific Heat**

Recall that, in statistical mechanics, the specific heat C is given by:

$$C = \frac{dE}{dT}.$$

Definition [specific heat] For each $m \in \mathbb{N}^+$ and each real $T > 0$, $C_m(T)$ of finite size m is defined as

$$C_m(T) := \frac{d}{dT} E_m(T) = \frac{\ln 2}{T^2} \left\{ \frac{\sum_{i=1}^m |q_i|^2 2^{-|q_i|/T}}{\sum_{i=1}^m 2^{-|q_i|/T}} - \left(\frac{\sum_{i=1}^m |q_i| 2^{-|q_i|/T}}{\sum_{i=1}^m 2^{-|q_i|/T}} \right)^2 \right\}.$$

Then, for each $T > 0$, the specific heat $C(T)$ is defined as

$$C(T) := \lim_{m \rightarrow \infty} C_m(T).$$

Theorem [specific heat] Let $T \in \mathbb{R}$.

(i) If $0 < T < 1$ and T is **computable**, then $C(T)$ converges to a left-computable real which is Chaitin T -random and T -compressible.

\Rightarrow The partial randomness of $C(T)$ equals to the temperature T .

(ii) If $T = 1$, then $C(T)$ diverges to ∞ .

\Rightarrow "Phase Transition" occurs at temperature 1.

In the case of $T > 1$, it is still open whether $C(T)$ diverges or not.

Thermodynamic Quantities in AIT: **Properties in Summary**

Definition We say $\alpha \in \mathbb{R}$ is weakly Chaitin T -random if $H(\alpha \upharpoonright n) - Tn \geq O(1)$.

Theorem [partition function $Z(T)$, free energy $F(T)$] Let $T \in \mathbb{R}$.

- (i) If $0 < T \leq 1$ and T is **computable**, then each of $Z(T)$ and $F(T)$ converges to real which is **weakly** Chaitin T -random and T -compressible.
- (ii) If $1 < T$, then $Z(T)$ and $F(T)$ diverge to ∞ and $-\infty$, respectively. \square

Definition We say $\alpha \in \mathbb{R}$ is Chaitin T -random if $\lim_{n \rightarrow \infty} H(\alpha \upharpoonright n) - Tn = \infty$.

Theorem [energy $E(T)$, entropy $S(T)$, specific heat $C(T)$] Let $T \in \mathbb{R}$.

- (i) If $0 < T < 1$ and T is **computable**, then each of $E(T)$, $S(T)$, and $C(T)$ converges to real which is Chaitin T -random and T -compressible.
- (ii) If $1 \leq T$, then both $E(T)$ and $S(T)$ diverge to ∞ . In the case of $T = 1$, $C(T)$ diverge to ∞ . \square

Implication of (i): The partial randomness of the values of the thermodynamic quantities equals to the temperature T .

Thermodynamic Interpretation of (ii): “Phase Transition” occurs at temperature 1.

Thermodynamic Quantities in AIT: **Temperature**

Temperature \Rightarrow **Fixed Point Theorems**

In the case where T is computable with $0 < T < 1$,
all of the partial randomness of the thermodynamic quantities:
partition function $Z(T)$, free energy $F(T)$,
energy $E(T)$, entropy $S(T)$, and specific heat $C(T)$,
equal to the temperature T .

However,
one of the most typical thermodynamic quantities is temperature T itself.

Thus, the following question arises naturally:

Question Can the partial randomness of the temperature equal to
the temperature itself ? Self-referential Question

We can answer this question affirmatively in the following form:

Fixed Point Theorem on Partial Randomness: **Main Theorem**

Theorem [fixed point theorem on partial randomness, Tadaki, CiE 2008]

For every $T \in (0, 1)$, if $Z(T)$ is a computable real, then

(i) T is weakly Chaitin T -random and T -compressible, and therefore

\Rightarrow The partial randomness of T equals to T itself.

(ii) $\lim_{n \rightarrow \infty} \frac{H(T \upharpoonright n)}{n} = T$.

\Rightarrow The compression rate of T equals to T itself.

Intuitive Meaning; Metaphor

Consider a file of infinite size whose content is

“The compression rate of this file is 0.100111001.....”

When this file is compressed, the compression rate of this file actually equals to 0.100111001....., as the content of this file says.

This situation forms a fixed point and is self-referential !

Fixed Point Theorem on Partial Randomness: **Proof**

Proof of Fixed Point Theorem

Theorem [fixed point theorem on partial randomness] [posted again]

For every $T \in (0, 1)$, if $Z(T)$ is a **computable** real, then

(i) T is right-computable and not left-computable,

(ii) T is weakly Chaitin T -random and T -compressible,

(iii) $\lim_{n \rightarrow \infty} \frac{H(T \upharpoonright n)}{n} = T$ (and therefore T is not computable). □

Lemma [upper bound I] For every $T \in (0, 1)$, if $Z(T)$ is right-computable then T is also right-computable. □

Lemma [upper bound II] For every $T \in (0, 1)$, if $Z(T)$ is left-computable and T is right-computable, then T is T -compressible. □

Lemma [lower bound] For every $T \in (0, 1)$, if $Z(T)$ is right-computable then T is weakly Chaitin T -random. □

Proofs of the three lemmas

Lemma [upper bound I] For every $T \in (0, 1)$, if $Z(T)$ is right-computable then T is also right-computable.

Proof) For each $k \in \mathbb{N}^+$ and each $x \in (0, 1)$, let $\omega_k(x) = \sum_{i=1}^k 2^{-|p_i|/x}$, where p_1, p_2, p_3, \dots is a particular recursive enumeration of $\text{Dom } U$.

Then we see that, for every $r \in \mathbb{Q} \cap (0, 1)$, $T < r$ if and only if there exists $k \in \mathbb{N}^+$ such that $Z(T) < \omega_k(r)$. This is because $Z(x) = \sum_{i=1}^{\infty} 2^{-|p_i|/x}$ is an increasing function of $x \in (0, 1]$ and $\lim_{k \rightarrow \infty} \omega_k(r) = Z(r)$.

Since $Z(T)$ is right-computable,

the set $\{r \in \mathbb{Q} \cap (0, 1) \mid \exists k \in \mathbb{N}^+ Z(T) < \omega_k(r)\}$ is r.e. and therefore

the set $\{r \in \mathbb{Q} \cap (0, 1) \mid T < r\}$ is also r.e. □

Lemma [upper bound II] For every $T \in (0, 1)$, if $Z(T)$ is left-computable and T is right-computable, then T is T -compressible.

Proof) Omitted. □

Lemma [lower bound] For every $T \in (0, 1)$, if $Z(T)$ is right-computable then T is weakly Chaitin T -random.

Proof) The following procedure calculates a partial recursive function $\Psi: \{0, 1\}^* \rightarrow \{0, 1\}^*$ such that $Tn - Tc < H(\Psi(T \upharpoonright n))$. The lemma follows from $H(\Psi(T \upharpoonright n)) \leq H(T \upharpoonright n) + O(1)$. Let $\omega_k(x) = \sum_{i=1}^k 2^{-|p_i|/x}$.

Procedure: Given $T \upharpoonright n$, one can effectively find k_0 which satisfies

$$Z(T) < \omega_{k_0}(0.(T \upharpoonright n) + 2^{-n}).$$

This is possible because $Z(x)$ is an increasing function of x , $\lim_{k \rightarrow \infty} \omega_k(r) = Z(r)$ for every $r \in \mathbb{Q} \cap (0, 1)$, and $Z(T)$ is right-computable. It follows that

$$\sum_{i=k_0+1}^{\infty} 2^{-\frac{|p_i|}{T}} = Z(T) - \omega_{k_0}(T) < \omega_{k_0}(0.(T \upharpoonright n) + 2^{-n}) - \omega_{k_0}(T) < 2^{c-n}.$$

Hence, for every $i > k_0$, $2^{-\frac{|p_i|}{T}} < 2^{c-n}$ and therefore $Tn - Tc < |p_i|$. Thus, by calculating the set $\{U(p_i) \mid i \leq k_0\}$ and picking any one finite binary string s which is not in this set, one can then obtain $s \in \{0, 1\}^*$ such that $Tn - Tc < H(s)$. □

Remark on the sufficient condition in the fixed Point Theorem

Theorem [fixed point theorem on partial randomness] [posted again]

For every $T \in (0, 1)$, if $Z(T)$ is computable, then T is weakly Chaitin T -random and T -compressible. \square

Note that $Z(x) = \sum_{i=1}^{\infty} 2^{-|q_i|/x}$ is a strictly increasing continuous function of $x \in (0, 1)$, and the set of all computable reals is dense in \mathbb{R} . Thus,

Theorem The set $\{T \in (0, 1) \mid Z(T) \text{ is computable}\}$ is dense in $(0, 1)$. \square

Corollary [density of the fixed points]

The set $\{T \in (0, 1) \mid T \text{ is weakly Chaitin } T\text{-random and } T\text{-compressible}\}$ is dense in $(0, 1)$. \square

At this point, the following question would arise naturally:

Question Is this sufficient condition, i.e., **the computability of $Z(T)$** , also necessary for T to be a fixed point ?

Answer Completely not !! (as we can see through the following argument)

Thermodynamic Quantities in AIT: **Fixed Point Theorems**

In the fixed point theorem, $Z(T)$ can be replaced by each of the thermodynamic quantities $F(T)$, $E(T)$, and $S(T)$ as follows.

Theorem [fixed point theorem by the free energy $F(T)$, Tadaki 2009]

For every $T \in (0, 1)$, if $F(T)$ is computable, then

- (i) T is weakly Chaitin T -random and T -compressible, and therefore
- (ii) $\lim_{n \rightarrow \infty} H(T \upharpoonright n)/n = T$. □

This fixed point theorem has the exactly same form as by $Z(T)$.

Theorem [fixed point theorem by the energy $E(T)$, Tadaki 2009]

For every $T \in (0, 1)$, if $E(T)$ is computable, then

- (i) T is Chaitin T -random and T -compressible, and therefore
- (ii) $\lim_{n \rightarrow \infty} H(T \upharpoonright n)/n = T$. □

Theorem [fixed point theorem by the entropy $S(T)$, Tadaki 2009]

For every $T \in (0, 1)$, if $S(T)$ is computable, then

- (i) T is Chaitin T -random and T -compressible, and therefore
- (ii) $\lim_{n \rightarrow \infty} H(T \upharpoonright n)/n = T$. □

Proofs of the fixed point theorems by $F(T)$, $E(T)$, and $S(T)$

Lemma [thermodynamic relations] $T \in (0, 1)$.

(i) $F'_k(T) = -S_k(T)$, $E'_k(T) = C_k(T)$, and $S'_k(T) = C_k(T)/T$.

(ii) $F'(T) = -S(T)$, $E'(T) = C(T)$, and $S'(T) = C(T)/T$.

(iii) $S_k(T), C_k(T) \geq 0$. $S_k(T), C_k(T) > 0$ for all sufficiently large k .
 $S(T), C(T) > 0$. □

A portion of the proof of the above lemma :

In particular, the condition (iii) is proved based on **the statistical mechanical relations**:

There exists $k_0 \in \mathbb{N}^+$ such that, for every $k \geq k_0$ and every $T \in (0, 1)$,

$$S_k(T) = - \sum_{i=1}^k \frac{2^{-\frac{|p_i|}{T}}}{Z_k(T)} \log_2 \frac{2^{-\frac{|p_i|}{T}}}{Z_k(T)} > 0, \quad \text{Shannon entropy of } \left\{ \frac{2^{-\frac{|p_i|}{T}}}{Z_k(T)} \right\}_i$$

$$C_k(T) = \frac{\ln 2}{T^2} \sum_{i=1}^k \{ |p_i| - E_k(T) \}^2 \frac{2^{-\frac{|p_i|}{T}}}{Z_k(T)} > 0. \quad \text{Variance of energy } E_k(T)$$

Relation between the sufficient conditions of FPTs

Theorem There does not exist $T \in (0, 1)$ such that both $Z(T)$ and $F(T)$ are computable.

Proof)

Contrarily, assume that both $Z(T)$ and $F(T)$ are computable for some $T \in (0, 1)$. Since the statistical mechanical relation $F(T) = -T \log_2 Z(T)$ holds,

$$T = -\frac{F(T)}{\log_2 Z(T)}.$$

Thus, T is computable, and therefore $Z(T)$ is weakly Chaitin T -random, i.e., $Tn \leq H(Z(T) \upharpoonright n) + O(1)$. However, this is impossible, since $Z(T)$ is computable by the assumption, and therefore $H(Z(T) \upharpoonright n) \leq 2 \log_2 n + O(1)$. Thus we have a contradiction. \square

$$\{T \in (0, 1) \mid Z(T) \text{ is computable}\} \cap \{T \in (0, 1) \mid F(T) \text{ is computable}\} = \emptyset.$$

dense in $(0, 1)$ dense in $(0, 1)$

In particular, this shows that the computability of $Z(T)$ is not a necessary condition for T to be a fixed point in the fixed point theorem by $Z(T)$.

Summary

Partial Randomness = Temperature.
(Compression Rate)

Remark: Mathematical Implication of the Results

The proofs of the fixed point theorems on partial randomness depend heavily on the following thermodynamic relations:

Lemma [thermodynamic relations] $T \in (0, 1)$.

(i) $F'_k(T) = -S_k(T)$, $E'_k(T) = C_k(T)$, and $S'_k(T) = C_k(T)/T$.

(ii) $F'(T) = -S(T)$, $E'(T) = C(T)$, and $S'(T) = C(T)/T$.

(iii) $S_k(T), C_k(T) \geq 0$. $S_k(T), C_k(T) > 0$ for all sufficiently large k .
 $S(T), C(T) > 0$. □

Moreover, the proof of the following theorem depends on the statistical mechanical relation $F(T) = -T \log_2 Z(T)$.

Theorem There does not exist $T \in (0, 1)$ such that both $Z(T)$ and $F(T)$ are computable. □

This theorem says that the computability of $F(T)$ gives completely different fixed points from the computability of $Z(T)$.

These fact would imply that **the analytic method can be used in the research of AIT (algorithmic randomness)**.

Remark: Physical Implication of the Results

Definition Let q_1, q_2, q_3, \dots be an arbitrary enumeration of $\text{Dom } U$. \square

In the statistical mechanical interpretation of AIT,

q_1, q_2, q_3, \dots correspond to energy eigenstates of a quantum system and $|q_1|, |q_2|, |q_3|, \dots$ correspond to energy eigenvalues of the quantum system with degeneracy.

Theorem [distribution of programs (i.e., “energy eigenstates”), Solovay]

$$\#\{p \mid p \in \text{Dom } U \ \& \ |p| \leq n\} = 2^{n-H(n)+O(1)} \text{ for all } n \in \mathbb{N}.$$

(In statistical mechanics, this quantity represents “the number of states below energy n ”)

Here $H(n) = H(\text{the base-two representation of } n)$. \square

If the energy eigenvalues of a quantum system distribute according to the above distribution, then the following situation can realize:

If T is a computable real, the partial randomness of the values of thermodynamic quantities at temperature T equals to T in the quantum system.

Appendix

Proof of the fixed point theorem by energy $E(T)$

Theorem [general form of fixed point theorem] Let $f: (0, 1) \rightarrow \mathbb{R}$.

Suppose that f is a strictly increasing function and there is $g: (0, 1) \times \mathbb{N}^+ \rightarrow \mathbb{R}$ which satisfies the following conditions:

(i) $\forall T \in (0, 1) \lim_{k \rightarrow \infty} g(T, k) = f(T)$.

(ii) The mapping $(\mathbb{Q} \cap (0, 1)) \times \mathbb{N} \ni (r, k) \mapsto g(r, k)$ is computable.

(iii) $\forall T \in (0, 1) \exists k_0 \in \mathbb{N}^+ \exists a \in \mathbb{N}^+ \exists b, c, d \in \mathbb{N} \forall k \geq k_0$

$$\left|p_{k+1}\right|^a 2^{-\frac{|p_{k+1}|}{T}-b} \leq g(T, k+1) - g(T, k) \leq \left|p_{k+1}\right|^c 2^{-\frac{|p_{k+1}|}{T}+d}.$$

(iv) $\forall T \in (0, 1) \exists t \in (T, 1) \exists k_0 \in \mathbb{N}^+ \exists c, d \in \mathbb{N} \forall k \geq k_0 \forall x \in (T, t)$

$$2^{-c}(x - T) \leq g(x, k) - g(T, k) \leq 2^d(x - T).$$

(v) $\forall t_1, t_2 \in (0, 1)$ with $t_1 < t_2 \exists k_0 \in \mathbb{N}^+ \forall k \geq k_0 \forall x \in [t_1, t_2] g(x, k) \leq f(x)$.

(vi) $\forall k \in \mathbb{N}^+ \forall T \in (0, 1) \lim_{x \rightarrow T+0} g(x, k) = g(T, k)$.

Then, for every $T \in (0, 1)$, if $f(T)$ is computable, then T is Chaitin T -random and T -compressible. □

Proof of the fixed point theorem by energy $E(T)$

Theorem [fixed point theorem by energy $E(T)$] [posted again]

For every $T \in (0, 1)$, if $E(T)$ is computable, then T is Chaitin T -random and T -compressible. \square

A portion of the proof to confirm the condition (iv) :

Using the mean value theorem and the lemma below, for every $k \in \mathbb{N}^+$ and every $T, x, t \in (0, 1)$ with $T < x < t$, there exists $y \in (T, x)$ such that

$$E_k(x) - E_k(T) = C_k(y)(x - T).$$

On the other hand, for every $T_1, T_2 \in (0, 1)$ with $T_1 < T_2$, there exist $a \in \mathbb{N}$ and $k_0 \in \mathbb{N}^+$ such that, every $k \geq k_0$ and every $y \in [T_1, T_2]$,

$$0 < \min C_{k_0}([T_1, T_2]) \leq C_k(y) < \max C([T_1, T_2]).$$

Lemma [thermodynamic relations] $T \in (0, 1)$.

(i) $F'_k(T) = -S_k(T)$, $E'_k(T) = C_k(T)$, and $S'_k(T) = C_k(T)/T$.

(ii) $F'(T) = -S(T)$, $E'(T) = C(T)$, and $S'(T) = C(T)/T$.

(iii) $S_k(T), C_k(T) \geq 0$. $S_k(T), C_k(T) > 0$ for all sufficiently large k .
 $S(T), C(T) > 0$. \square

Relation between the sufficient conditions of FPTs II

Theorem There does not exist $T \in (0, 1)$ such that all of $Z(T)$, $E(T)$, and $S(T)$ are computable.

Proof) Use the statistical mechanical relation

$$S(T) = \frac{E(T)}{T} + \log_2 Z(T).$$

□

Theorem There does not exist $T \in (0, 1)$ such that all of $F(T)$, $E(T)$, and $S(T)$ are computable.

Proof) Use the thermodynamic relation

$$S(T) = \frac{E(T) - F(T)}{T}.$$

□

Some other property of the sufficient condition in FPTs

Using the fixed point theorem by $Z(T)$, some property of the computability of $Z(T)$ is derived.

Let $T \in (0, 1)$ and $a \in (0, 1]$. Assume that a is computable.

$$Z(aT) \text{ is computable} \Rightarrow \lim_{n \rightarrow \infty} \frac{H((aT) \upharpoonright n)}{n} = aT \Rightarrow \lim_{n \rightarrow \infty} \frac{H(T \upharpoonright n)}{n} = aT.$$

by FPT by $H((aT) \upharpoonright n) = H(T \upharpoonright n) + O(1)$

Theorem $S_a \cap S_b = \emptyset$ for any distinct computable reals $a, b \in (0, 1]$, where $S_a = \{T \in (0, 1) \mid Z(aT) \text{ is computable}\}$. □

Example For every $T \in (0, 1)$, if $Z(T)$ is computable, then for each integer $n \geq 2$, $Z(T/n)$ is **not** computable. **Namely,**

for every $T \in (0, 1)$, if the sum $\sum_{p \in \text{Dom } U} 2^{-|p|/T}$ is computable, then its power sum $\sum_{p \in \text{Dom } U} (2^{-|p|/T})^n$ is **not** computable for every integer power $n \geq 2$. □