Algorithmic randomness and monotone complexity on product space

Hayato Takahashi

The Institute of Statistical Mathematics

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Conditional probability:

P: computable probability on $X \times Y$ $(= \Omega^2)$.
marginal distributions

$$P_X(x) := P(x, \Omega).$$

$$P_Y(y) := P(\Omega, y).$$

conditional probability :

$$P(x|y) := P(x, y)/P_Y(y) \text{ for } y \in S$$

$$P(x|y^\infty) := \lim_{y \to y^\infty} P(x, y)/P_Y(y) \text{ for } y^\infty \in \Omega$$

(if the right-hand-side exists.)

Notation:

$S := \{0, 1\}^*$, $\Omega := \{0, 1\}^\infty$, $x, y \in S$, $x^\infty, y^\infty \in \Omega$.

$\Delta(x) := \{xy^\infty | y^\infty \in \Omega\}$, $P(x, y) := P(\Delta(x), \Delta(y))$

$\mathcal{R}^P$: the set of ML random sequences w.r.t. P.

Theorem 1 *If* $y^\infty \in \mathcal{R}^{P_Y}$ *then* $P(\cdot|y^\infty)$ *is a probability measure on* Ω*.*

Remark: It is known that $P(\cdot|y^\infty)$ is a probability measure on Ω for almost all y^∞.

Outline of proof)

- $\lim_{y\to y^\infty} P(x,y)/P_Y(y) \leq 1$ exists by martingale convergence theorem for random sequences.

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$$\begin{aligned} \forall x,\ 0 &= \lim_{y\to y^\infty} P(x|y) - P(x0|y) - P(x1|y) \\ &= P(x|y^\infty) - P(x0|y^\infty) - P(x1|y^\infty) \end{aligned}$$

qed

Lambalgen's theorem for correlated probability:

$$\mathcal{R}^P_{y^\infty} := \{x^\infty \in \Omega | (x^\infty, y^\infty) \in \mathcal{R}^P\}.$$

$$P(\cdot|y^\infty):\ \text{conditional prob. w.r.t. } y^\infty.$$

$$\mathcal{R}^{P(\cdot|y^\infty),y^\infty} :=\ \text{the set of ML random sequences w.r.t.} P(\cdot|y^\infty).$$

Theorem 2 *Let P be a computable probability on $X \times Y$ $(= \Omega^2)$. If $P(\cdot\,|\,y^\infty)$ is computable relative to $y^\infty \in \mathcal{R}^{P_Y}$, then*

$$\mathcal{R}^{P(\cdot\,|\,y^\infty),\,y^\infty} = \mathcal{R}^P_{y^\infty}.$$

In [Takahashi 2008], "uniform computability" assumption was used to show $\mathcal{R}^{P(\cdot\,|\,y^\infty),\,y^\infty} \supset \mathcal{R}^P_{y^\infty}$.

Notation: $S' \subset S$, $\tilde{S}' := \cup_{s \in S'} \Delta(s)$.

Outline of $\mathcal{R}^{P(\cdot\,|\,y^\infty),\,y^\infty} \supset \mathcal{R}^P_{y^\infty}$:

The proof is done by extending a ML-test w.r.t. $P(\cdot\,|\,y^\infty)$ to a ML-test w.r.t. P on Ω^2.

Fix $y^\infty \in \mathcal{R}^P_Y$.

A, B : partial comp. func.

$A : S \times S \times \mathbb{N} \to \mathbb{Q}$, $\forall x, k \exists y \sqsubset y^\infty, |P(x|y^\infty) - A(x, y, k)| < \frac{1}{k}$.

$B : \mathbb{N} \times \mathbb{N} \times S \to S$, $U_n^{y^\infty} := \{B(i, n, y) | \exists i, y \sqsubset y^\infty\}$ is a ML-test w.r.t. $P(\cdot\,|\,y^\infty)$ and $P(\tilde{U}_n^{y^\infty} | y^\infty) < 2^{-n}$.

For simplicity, suppose that $P(x|y^\infty) > 0$ for all x.

$$V_n := \{(x,y,k) | \exists i, B(i,n,y) = x,\ \frac{1}{k} < \frac{1}{2} A(x,y,k)\}.$$

Since $P(x|y^\infty) > 0$, V_n is r.e. and $P(x|y^\infty) < \frac{3}{2} A(x,y,k)$.

Then there is a r.e. $W_n \subset S \times S \times \mathbb{N}$ s.t.

$\tilde{W'}_n \subset \tilde{V'}_n,$ where $W'_n = \{(x,y) | (x,y,k) \in W_n\},\ V'_n = \{(x,y) | (x,y,k) \in V_n\}$,

$$\sum_{(x,y,k) \in W_n} A(x,y,k) P(y) < 2^{-n+1},$$

$$\tilde{U}_n^{y^\infty} = \tilde{W'}_{n,y^\infty}.$$

Finally let

$$U_n := \{(x,z) \in S \times S \mid \quad (x,y) \in W'_n,\ y \sqsubseteq z, \\ P(x,y) < \frac{3}{2} A(x,y,k) P(y)\}.$$

Then $P(\tilde{U}_n) < \frac{3}{2} \sum_{(x,y) \in W'_n} A(x,y,k) P(y) < 2^{-n+2}$.

Since $P(x|y) \to P(x|y^\infty)$ as $y \to y^\infty$ for $y^\infty \in \mathcal{R}^{P_Y}$,

$$\tilde{U}_n^{y^\infty} = \tilde{U}_{n,y^\infty},$$

and U_n is a ML-test of P.

The converse inclusion is shown in [takahashi 2008]. ■

Monotone complexity

Order structure:

$\Delta(x) := \{(\omega_1, \omega_2, \ldots) \in \Omega^\infty | \forall i, x_i \sqsubseteq \omega_i\}$ for $x = (x_1, x_2, \ldots) \in (S \cup \Omega)^\infty$.

$$x \sqsubseteq y \text{ if } \Delta(x) \supset \Delta(y).$$

$$((S \cup \Omega)^\infty, \sqsubseteq) : \text{ partially ordered set.}$$

$$S^* := \cup_k S^k, \ \eta : S^* \to S^\infty,$$

$$\eta(x) := (x, \lambda, \lambda, \ldots)$$

$$[x] := \{y | \eta(x) = \eta(y), y \in S^*\}$$

$$[x] \sqsubseteq [y] \text{ if } \eta(x) \sqsubseteq \eta(y).$$

$$S^*/ker(\eta) := \{[x] | x \in S^*\} : \text{ partially ordered set}$$

$\eta : S^*/ker(\eta) \to (S \cup \Omega)^\infty, \ \eta([x]) := \eta(x)$: 1-1 order preserving.

$\bigvee A$: the least upper bound of A for $A \subset S^*/ker(\eta)$.
The $\bigvee A$ exists in $(S \cup \Omega)^\infty$ iff $\cap_{x \in A} \Delta(\eta(x)) \neq \emptyset$.

Monotone function and complexity:

Let $F \subset S^*/ker(\eta) \times S^*/ker(\eta)$ and $F_p := \{x | (p, x) \in F\}$.
Assume:
a0) $\forall p \in S^*/ker(\eta),\ [\lambda] \in F_p$.
a1) $\forall p \in S^*/ker(\eta),\ \bigvee_{p' \sqsubseteq p} F_{p'}$ exists.

Set

$$f(p) := \bigvee_{p' \sqsubseteq p,\ p' \in S^*/ker(\eta)} F_{p'} \text{ for } p \in (S \cup \Omega)^\infty. \quad (1)$$

Then

$$f : (S \cup \Omega)^\infty \to (S \cup \Omega)^\infty \text{ and } p' \sqsubseteq p \Rightarrow f(p') \sqsubseteq f(p).$$

Conversely, let $f : (S \cup \Omega)^\infty \to (S \cup \Omega)^\infty$ be a monotone function, and set

$$F := \{(p, x) \in S^* \times S^* | \eta(x) \sqsubseteq f(\eta(p))\},$$

Then $\bigvee F_p = f(p)$,
F satisfies a0 and a1, and the function defined by F coincides with f.

If F is a r.e. set and F satisfies a0 and a1, then the function f defined by (1) is called *computable monotone function*.

$$Km_f(x) := \min\{|p| \,|\, x \sqsubseteq f(p)\}, \text{ for } x \in (S \cup \Omega)^\infty, |p| := \sum_i |p_i|.$$

$$Km_f(x) := Km_f(\eta(x)), \text{ for } x \in S^*.$$

$$Km(x) := Km_u(x), \ u \text{ is optimal}.$$

Levin-Schnorr theorem for product space:

$$\mathcal{A} \subset S^* \text{ and } r.e..$$

Condition 1) if $x, y \in \mathcal{A}$ then, x and y are comparable or

$$\Delta(x) \cap \Delta(y) = \emptyset.$$

Condition 2) there is a recursive function $f : S^* \times \mathbb{N} \to \mathcal{A}$ such that for any $x \in S^*$, $\Delta(x) = f(x, \mathbb{N})$ and $f(x, \mathbb{N})$ is prefix-free.

$$\mathcal{A}(x^\infty) := \{x \in S^* | x \in \mathcal{A},\ x \sqsubset x^\infty\}.$$

$$P: \text{ computable probability on } (\mathcal{B}_{\Omega^\infty}, \Omega^\infty).$$

$$\mathcal{B}_{\Omega^\infty} = \sigma\{\Delta(\eta(x)) | x \in S^*\}.$$

Theorem 3 (Levin-Schnorr theorem on product space) *Under Condition 1 and 2,*

$$x^\infty \in \mathcal{R}^P \text{ iff } \sup_{x \in \mathcal{A}(x^\infty)} -\log P(x) - Km(x) < \infty.$$

If $u : (S \cup \Omega)^\infty \to (S \cup \Omega)^\infty$ optimal and $f : S \cup \Omega \to (S \cup \Omega)^\infty$ then

$$Km_u(x) \leq Km_f(x) + O(1).$$

Corollary 1 (One dimensional coding)

If $\mathcal{A} \subset S^*$ *is* $decidable$

then there is $f : S \cup \Omega \to (S \cup \Omega)^\infty$ *and*

$$Km_f(x) \leq -\log P(x) + O(1) \text{ for } x \in \mathcal{A}.$$

In particular if $x^\infty \in \mathcal{R}^P$ *and* $\mathcal{A}$ *is decidable then under Condition 1 and 2,*

$$\sup_{x \in \mathcal{A}(x^\infty)} |Km_u(x) - Km_f(x)| < \infty.$$

Consistency of posterior distribution:

Example: Let P_θ be Bernoulli model, then $\lim_n E(\theta|X_1, X_2, \ldots X_n) = \theta,\ P_\theta - a.s.$ (consistency of posterior distribution).

We have a stronger statement for the consistency of posterior distribution.

Theorem 4 *Let* P *be a computable probability on* $X \times Y$*, where* $X = Y = \Omega$*. Assume that* $P_Y(y) > 0$ *for all* $y \in S$*. The following six statements are equivalent:*

(1) $P(\cdot \,|\, y) \perp P(\cdot \,|\, z)$ *if* $\Delta(y) \cap \Delta(z) = \emptyset$ *for* $y, z \in S$*.*

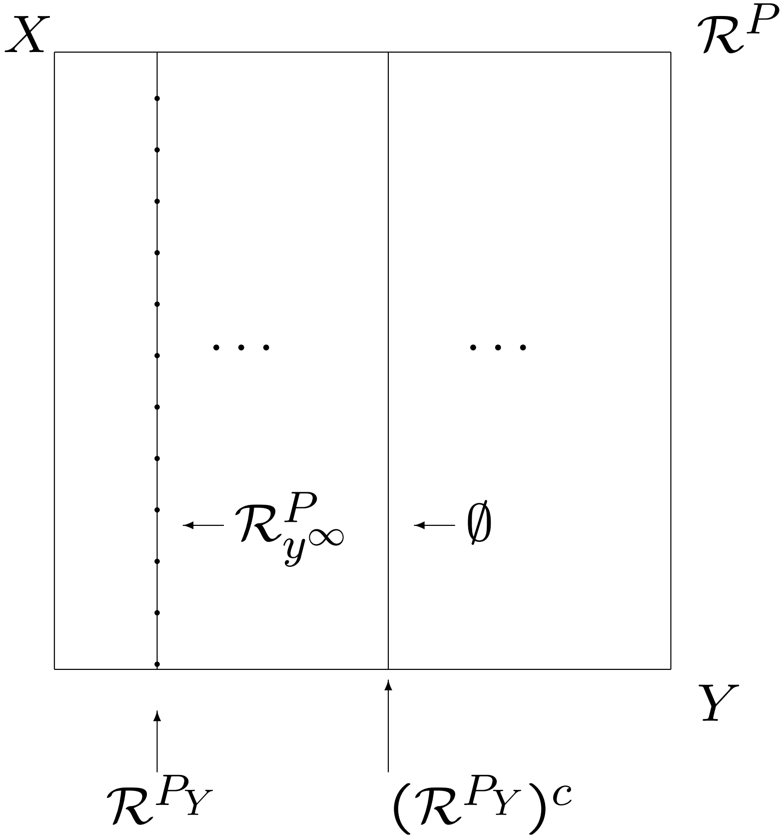
(2) $\mathcal{R}^{P(\cdot \,|\, y)} \cap \mathcal{R}^{P(\cdot \,|\, z)} = \emptyset$ *if* $\Delta(y) \cap \Delta(z) = \emptyset$ *for* $y, z \in S$*.*

(3) $P_{Y|X}(\cdot \,|\, x)$ *converges weakly to* I_{y^∞} *as* $x \to x^\infty$ *for* $(x^\infty, y^\infty) \in \mathcal{R}^P$*, where* I_{y^∞} *is the probability that has probability of 1 at* y^∞*.*

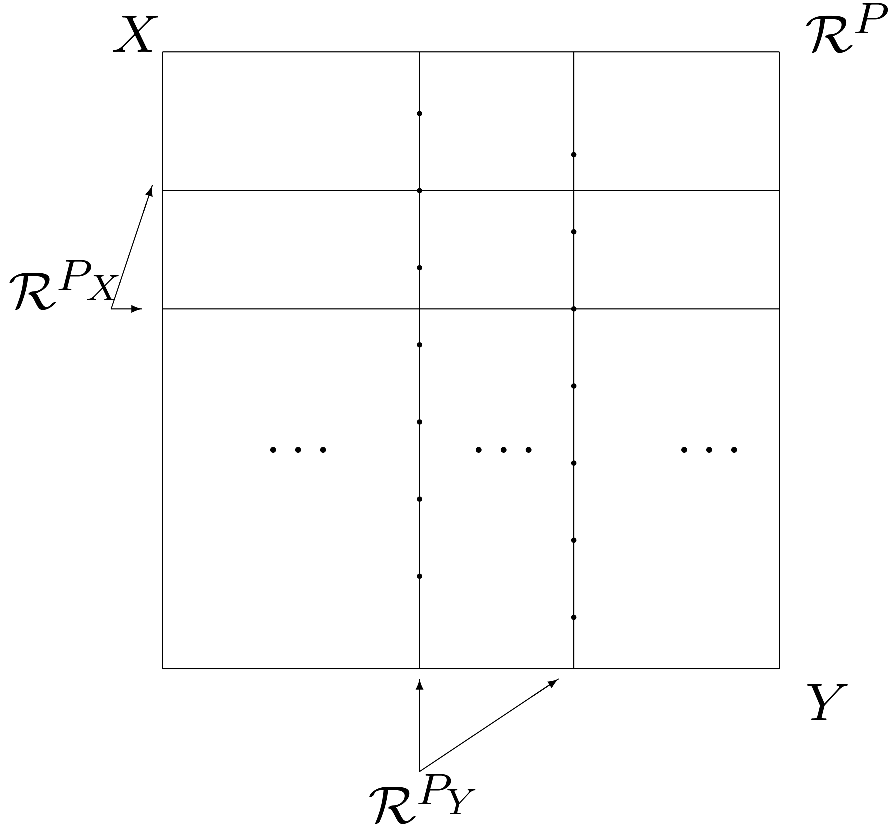
(4) $\mathcal{R}^P_{y^\infty} \cap \mathcal{R}^P_{z^\infty} = \emptyset$ *if* $y^\infty \neq z^\infty$*.*

(5) There exists $f : X \to Y$ *such that* $f(x^\infty) = y^\infty$ *for* $(x^\infty, y^\infty) \in \mathcal{R}^P$*.*

(6) There exists $f : X \to Y$ *and* $Y' \subset Y$ *such that* $P_Y(Y') = 1$ *and* $f = y^\infty, \; P(\cdot\,; y^\infty) - a.s.$ *for* $y^\infty \in Y'$*.*



General



Consistent

References:

Hayato Takahashi, On a definition of random sequences with respect to conditional probability, Information and Computation, 2008.

Thank you.