Some questions on higher randomness theory

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Notions on higher randomness

Definition

1. Given a class Γ of sets of reals, a real x is Γ if for every null set $X \in \Gamma$, $x \notin X$.
2. Given a class Γ of sets of reals , a real x is Γ-Martin-Löf (or *ML*)-random if for every set $X \subseteq \omega \times 2^{<\omega}$ in Γ for which $\forall n \mu(\bigcup_{(n,\sigma)\in X}[\sigma]) < 2^{-n}$, $x \notin \bigcap_{n\in\omega} X_n$.

Δ^1_1 v.s. Σ^1_1

Theorem (Sacks)

Δ^1_1-ML-randomness=Δ^1_1-randomness=Σ^1_1-ML-randomness=Σ^1_1-randomness.

So Σ^1_1-randomness collapses.

Theorem (Sacks)

There is neither largest Δ^1_1-null set nor largest Σ^1_1-null set.

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Π^1_1-*ML*-randomness

Theorem (Hjorth and Nies)

1. *There is a universal Π^1_1-ML-test.*
2. *For every $x \geq_h \mathcal{O}$, there is a Π^1_1-ML-random real $z \equiv_h x$.*
3. *There is a left-Π^1_1-ML-random real.*

Theorem (Chong, Nies and Yu)

Π^1_1-*ML-randomness* $\subset$ Δ^1_1-*randomness.*

Π^1_1-randomness

Theorem (Sacks)

$\{x \mid x \geq_h \mathcal{O}\}$ is a Π^1_1-null set.

So Π^1_1-randomness $\subset$ Π^1_1-*ML*-randomness.

Proposition (Chong, Nies and Yu)

A Δ^1_1-random real x is Π^1_1-random if and only if $x \not\geq_h \mathcal{O}$.

Theorem (Kechris; Hjorth and Nies)

There is a largest Π^1_1-null set.

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Strongly Π^1_1-*ML*-randomness

Definition

A real x is strongly Π^1_1-*ML*-random if for every generalized Π^1_1-*ML*-test $\{U_n\}_{n\in\omega}$, $x \notin \bigcap_n U_n$.

Theorem (Yu)

No left Π^1_1-real can be strongly Π^1_1-ML-random. So strongly Π^1_1-randomness $\subset$ Π^1_1-ML-randomness and there is no universal generalized Π^1_1-ML-test.

Question

Π^1_1-randomness $\subset$ strongly Π^1_1-ML-randomness? If yes, is it true that for for every $x \geq_h \mathcal{O}$, there is a strongly Π^1_1-ML-random real $z \equiv_h x$?

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Summary

$\Delta_1^1(\mathcal{O})$-randomness $\subset$ Π_1^1-randomness $\subseteq$ strongly Π_1^1-*ML*-randomness $\subset$ Π_1^1-*ML*-randomness $\subset$ Δ_1^1-randomness $=$ Δ_1^1-*ML*-randomness.

$\leq_K$ v.s. higher randomness

Does $\leq_K$ still work for higher randomness?

Proposition (Yu)

If $x \leq_K y$ and x is Π^1_1-random, then y is Π^1_1-random.

$\leq_K$ v.s. higher randomness

Does $\leq_K$ still work for higher randomness?

Proposition (Yu)

If $x \leq_K y$ and x is Π^1_1-random, then y is Π^1_1-random.

Higher Kurtz randomness

Definition

x is Δ^1_1-dominated if for every function $f \leq_h x$, f is dominated by a hyperarithmetic function.

Proposition (Bjorn, Nies, Stephan, Yu)

1. *If $\omega_1^x = \omega_1^{\mathrm{CK}}$, then x is Δ^1_1-Kurtz random if and only if x is Π^1_1-Kurtz random.*
2. *x is Π^1_1-random if and only if x is Δ^1_1-Kurtz random and Δ^1_1-dominated.*

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Lowness property

A real x is low for Γ-random if every Γ-random is $\Gamma(x)$-random, similarly for Γ-*ML*-random.

Theorem (Hjorth and Nies)

A real x is low for Π^1_1-ML-random if and only if x is hyperarithmetic.

Theorem (Chong, Nies and Yu)

There is a non-hyperarithmetic real which is low for Δ^1_1-random.

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Lowness for Π^1_1-randomness

Proposition (Harrington, Nies, Slaman)

A real x is low for Π^1_1-random if and only if x is low for Δ^1_1-random and non-Π^1_1-random cuppable.

Question

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Question

Is there a non-hyperarithmetic real which is low for Π^1_1-random?

Δ^1_1-traceable and semi-traceable

Definition

1. x is Δ^1_1-semi-traceable if for every function $f \leq_h x$ there is a hyperarithmetic function h so that there are infinitely many m such that $h(m) = f(m)$;
2. x is Δ^1_1-traceable if for every function $f \leq_h x$ there is a hyperarithmetic function h so that for every n, there is some $m \in (2^n, 2^{n+1})$ such that $h(m) = f(m)$.

Proposition (Kjors-Hanssen, Nies, Stephan and Yu)

There is a Δ^1_1-dominated and semi-traceable real which is not Δ^1_1-traceable.

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Proposition (Kjors-Hanssen, Nies, Stephan and Yu)

There is a Δ^1_1-dominated and semi-traceable real which is not $\Delta^1_!$-traceable.

Characterizing lowness for Kurtz randomness

Theorem (Kjors-Hanssen, Nies, Stephan and Yu)

x is low for Δ^1_1-Kurtz-random if and only if x is Δ^1_1-dominated and semi-traceable.

Proposition (KNSY)

Every low for Δ^1_1-Kurtz-random real is low for Π^1_1-Kurtz-random.

Question

Is there a non-hyperarithmetic real low for Π^1_1-Kurtz random?

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Characterizing lowness for Δ^1_1-randomness

Theorem (Chong, NIes and Yu)

x is low for Δ^1_1-random if and only if x is Δ^1_1-traceable.

Proposition (Chong, Nies and Yu)

There are $2^{\aleph_0}$-many Δ^1_1-traceable reals.

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Thanks