Fundamenta Informaticae XX (2011) 1–12
IOS Press

Limit Sets of Stable and Unstable Cellular Automata

Alexis Ballier[*†]
Centro de Modelamiento Matemático
Universidad de Chile
Av. Blanco Encalada 2120, Piso 7
Santiago, Chile

Pierre Guillon[†]
CNRS &
Department of Mathematics
FI-20014 University of Turku
Finland

Jarkko Kari[†]
Department of Mathematics
FI-20014 University of Turku
Finland

Abstract. We construct a cellular automaton (CA) with a sofic and mixing limit set and then construct a stable CA with the same limit set, showing there exist subshifts that can be limit sets of both stable and unstable CAs, answering a question raised by A. Maass.

1. Introduction

A cellular automaton is a very simple yet rich model for complex systems: a large number of basic elements interact locally, in such a way that some global structure emerges. Such behaviors can be observed in diverse scientific fields such as molecular or cellular biology, physics, ecology and sociology. Many different points of view have been adopted to formalize this complexity, using methods of combinatorics, topology, ergodic theory, language theory and theory of computation.

The configuration of the system is the global description of the states taken by every basic element, and we can see it evolve as (discrete) time goes. We are interested in the configurations that we will be able to see arbitrarily late in time. Usually, they are not all the configurations; we forget here the transient

Address for correspondence: Department of Mathematics, FI-20014 University of Turku, Finland
[*]Research supported by the FONDECYT Postdoctorado Proyecto 3110088
[†]Research supported by the Academy of Finland Grant 131558

phenomena and emphasize what will be seen in the long term. This set of configurations is called the limit set.

The limit set of cellular automata has been widely studied since the 80s: it can be uncomputable [7, 8, 9], and actually all of its nontrivial properties are undecidable [10]. Some additional properties (on cardinality or link with dynamical concepts) were proved in [4]. It is an open question to understand which sets of configurations can be limit sets of cellular automata; this was asked in particular [13] with some partial results that we state further down. Also note that a generalized form of this question, concerning subshift attractors, has been addressed in [5].

One of the questions left unanswered in [13] was whether when looking at the limit set of some cellular automaton, we could figure out whether this was reached after a finite number of steps or in the limit. We prove here that this is not the case.

In Section 2, we introduce the notions, the problem and our main result. Then, in Section 3, we build a particular cellular automaton that reaches its limit set in an infinite time and, in Section 4, we build another cellular automaton that reaches the same limit set, but after a finite time.

2. Definitions and notations

Shifts and CA. Let Σ be a finite set called the *alphabet*. The *i-shift* σ^i is the application from $\Sigma^{\mathbb{Z}}$ onto itself defined by $\sigma^i(x)_j = x_{j+i}$. We denote σ^1 simply by σ. A *local rule* is an application from Σ^V into Γ, for a finite *neighborhood* $V \subset \mathbb{Z}$ and finite alphabets Σ and Γ. A *block map* φ is defined by a local rule $\delta : \Sigma^V \to \Gamma$ as an application from $\Sigma^{\mathbb{Z}}$ into $\Gamma^{\mathbb{Z}}$ in the following way: $\varphi(x)_i = \delta(\sigma^i(x)_{|V})$. A *cellular automaton* (CA for short) is a block map defined by a local rule where $\Sigma = \Gamma$, *i.e.*, a block map from $\Sigma^{\mathbb{Z}}$ into itself.

We can endow Σ with the discrete topology and obtain a compact space $\Sigma^{\mathbb{Z}}$: the set of bi-infinite words, or *configurations*, over Σ. The product topology obtained over $\Sigma^{\mathbb{Z}}$ is induced by the metric: $d(x, y) = 2^{-\min\{|i|, i \in \mathbb{Z}, x_i \neq y_i\}}$. By Tychonoff's theorem, this topology is compact. A *subshift* of $\Sigma^{\mathbb{Z}}$ is a closed (for the above topology) $\sigma-$invariant subset of $\Sigma^{\mathbb{Z}}$. The set $\Sigma^{\mathbb{Z}}$ is called a *fullshift*.

It is well known that the block maps are exactly the continuous $\sigma-$commuting applications between their respective spaces (proven in [6] for CAs). In particular, the image $\varphi(\mathbf{X})$ of a subshift $\mathbf{X}$ by a block map φ is a subshift. We say that $\varphi(\mathbf{X})$ is a *factor* of $\mathbf{X}$, and $\mathbf{X}$ is a *cover* of $\varphi(\mathbf{X})$. If two subshifts are factors of one another, we say that they are *weakly conjugate*. Weak conjugacy is an equivalence relation. These concepts justify that we may still call *block map* the restriction of some block map to some subshift.

Limit set. The *limit set* Ω_Δ of the cellular automaton Δ is the set of configurations which admit preimages arbitrarily early in time:

$$\Omega_\Delta = \bigcap_{n \in \mathbb{N}} \Delta^n(\Sigma^{\mathbb{Z}}) \ .$$

By compactness, the limit set is never empty; it is actually a subshift that always contains at least one uniform configuration (*i.e.*, with all cells having the same state). It is the fullshift $\Sigma^{\mathbb{Z}}$ if and only if Δ is onto.

Since the number of subshifts is uncountable and the number of cellular automata is countable, it is clear that we cannot obtain any subshift as the limit set of a cellular automaton and, therefore, the limit sets of CAs must fullfill some additional conditions. We may distinguish two classes of cellular automata: a CA Δ is said to be *stable* if it reaches its limit set in finite time, *i.e.*, there exists an integer N such that $\Omega_\Delta = \Delta^N(\Sigma^{\mathbb{Z}})$; Δ is said to be *unstable* otherwise. Note that in the former case, the limit set is a factor of the fullshift, via the block map Δ^N.

Language and entropy. For a subshift $\mathbf{X}$, we can define its language of words of length n by $\mathcal{L}_n(\mathbf{X}) = \{x_0x_1 \dots x_{n-1} \mid x \in \mathbf{X}\}$. The *language* of the subshift $\mathbf{X}$ is defined by $\mathcal{L}(\mathbf{X}) = \bigcup_{n\in\mathbb{N}} \mathcal{L}_n(\mathbf{X})$. One can define the *entropy* of a subshift, which roughly speaking represents the exponential growth rate of its language:

$$h(\mathbf{X}) = \lim_{n\to\infty} \frac{\log |\mathcal{L}_n(\mathbf{X})|}{n}$$

If $\mathbf{X}$ is a factor of $\mathbf{S}$ then $h(\mathbf{X}) \leq h(\mathbf{S})$.

Sofic and finite type. We can define a subshift with a directed labeled graph G: the labels on the edges take their values over the alphabet Σ; a bi-infinite path in G defines a configuration of $\Sigma^{\mathbb{Z}}$ by reading the labels over the edges. We call this set of configurations the *sofic subshift* defined by G, denoted by $\mathbf{X}_G$. It is easily verified that $\mathbf{X}_G$ is indeed a subshift [12, Theorem 3.1.4]. Sofic subshifts are exactly those whose language is regular, and one can see the graph defining a sofic subshift as the finite state automaton recognizing its language [12, Notes of Chapter 3].

A word $w \in \mathcal{L}(\mathbf{X}_G)$ is *synchronizing* for graph G if all paths labeled w end in the same vertex. If the graph G has the property that all paths of length $k \geq 1$ are synchronizing, then $\mathbf{X}_G$ is a *subshift of finite type* of *order* k (SFT for short).

If $\mathbf{X}_G \subset \Sigma^{\mathbb{Z}}$ is a sofic subshift corresponding to some graph G, we can relabel all edges so that all edges have distinct labels. The subshift $\mathbf{Y}$ that this new graph represents is an SFT of order 2. It is a cover of $\mathbf{X}_G$, with a block map π of neighborhood $\{0\}$ and $\pi(\mathbf{Y}) = \mathbf{X}_G$. The *Shannon cover* of a sofic subshift $\mathbf{X}$ is the cover defined in this way (up to letter renaming) from the graph corresponding to the minimal deterministic automaton (*i.e.*, one state per possibilities to extend words) of the language $\mathcal{L}(\mathbf{X})$. It has the same entropy as $\mathbf{X}$, and the graph always admits (at least) one synchronizing word, called a *magic* word for $\mathbf{X}$. Sofic subshift $\mathbf{X}$ is an SFT if and only if the corresponding block map is reversible (the subshift can be essentially seen as the set of edges up to some renaming), or equivalently, all sufficiently long words are magic.

We have just seen that sofic subshifts are factors of SFTs, and the converse is true [12, Theorem 3.2.1]. Since $\Sigma^{\mathbb{Z}}$ is an SFT (represented by a one-vertex graph), limit sets of stable CAs are sofic subshifts. On the other hand, it is known that an SFT cannot be the limit set of an unstable CA [9].

Irreducibilities. A subshift $\mathbf{X}$ is said to be *irreducible* if for any two words $u, v \in \mathcal{L}(\mathbf{X})$, there exists a word w such that $uwv \in \mathcal{L}(\mathbf{X})$. A sofic subshift is irreducible if and only if it can be defined by a strongly connected graph. A subshift $\mathbf{X}$ is said to be *mixing* if for any two words $u, v \in \mathcal{L}(\mathbf{X})$, there exists N such that for any integer $n \geq N$, there exists a word w of length n such that $uwv \in \mathcal{L}(\mathbf{X})$. A sofic subshift is mixing if and only if it can be defined by a graph that is strongly connected and whose cycle lengths are relatively prime. (See [12] for graph-theoretic characterizations of some dynamical properties of sofic subshifts). Irreducibility and mixingness are preserved by block maps, and by taking the Shannon cover. Since $\Sigma^{\mathbb{Z}}$ is mixing, limit sets of stable CAs are mixing subshifts.

Receptivity. A uniform configuration $x = \ldots 00000 \ldots$ of a sofic subshift $\mathbf{X}$ is *receptive* if there are magic words w and w' such that for any $i \in \mathbb{N}$, $w0^iw' \in \mathcal{L}(\mathbf{X})$. If $\mathbf{X}$ is an SFT then any uniform configuration is receptive. Otherwise, the existence of a receptive configuration can be seen as the existence of a loop on one vertex of the corresponding (connected) graph. It can be proven that if $\mathbf{X}$ is mixing and admits a receptive configuration, then so does any of its factors.

As a result, the following property, denoted $(\mathbf{H})$, is a necessary condition for being a factor of a fullshift: sofic, mixing, with a receptive configuration. Actually, $(\mathbf{H})$ was proven by M. Boyle to be also sufficient [2, Theorem 3.3]. Recall that limit sets of stable CAs are factors of fullshifts, and thus satisfy $(\mathbf{H})$. The question was asked by A. Maass whether this was also an equivalence. He actually proved that it was the case for a subclass of these subshifts (those that are, additionally, AFTs, *i.e.*, such that that the factor map π corresponding to the Shannon cover is left-closing – see Section 4 for the definition). He mentions two other open questions:

- Does there exist an unstable CA that has a limit set with property $(\mathbf{H})$?
- Does there exist a subshift that is both the limit set of a stable and an unstable CA?

A positive answer to the latter question implies a positive one to the former. Since A. Maass's work, some partial answers to these questions have been obtained: there exists an unstable CA [5] with a mixing (non sofic) limit set; if two CAs have the same limit set and the same dynamics over their limit set then they are either both stable or both unstable [11]. In this paper, we answer both questions: the first one in Section 3 by constructing an unstable CA with a limit set with property $(\mathbf{H})$. In Section 4, we construct a stable CA with the same limit set, allowing us to state our main result.

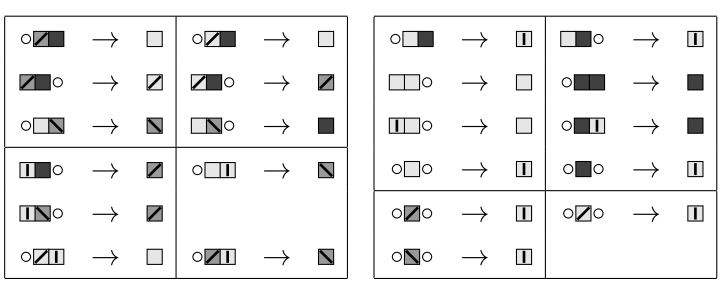
Proposition 2.1. There exists a subshift which is the limit set of both a stable and an unstable CA.

Proof:
The subshift $\mathbf{X}$ depicted on Figure 2 is the limit set of an unstable CA by Proposition 3.1 and of a stable CA by Proposition 4.2. □

3. Unstable CA

All the subshifts considered below will be defined on the same 6-letter alphabet: $\Sigma = \{$□, ■, ◪, ◩, ⧅, ◫$\}$.

Definition 3.1. Let U be the CA depicted on Figure 1, with neighborhood $\{-1, 0, 1\}$ and the following local rule, where the first matching rule is applied if any, otherwise the state is unchanged. In the rules, $\circ$ indicates a wild card that represents an arbitrary state:



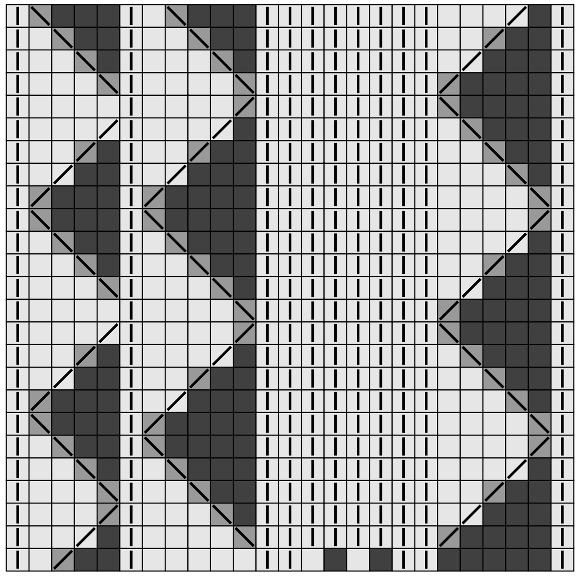


Figure 1. A space-time diagram of the unstable CA. Time increases upwards.

The idea behind the local rule defining U should be better understandable by looking at its space-time diagram on Figure 1 with the following explanation: we have barriers (◫) and signals bouncing between the barriers (▨, ⧄ and ⧅). We want to have at most one signal between two barriers so we add states saying "I am at the left of a signal" (□) and "I am at the right of a signal" (■). We want to have at least one signal between two barriers (if there is enough space of course), so if a cell believing there should be a signal on its right (□) is seeing, instead, a barrier (◫) on its right, the □ cell becomes a left-moving signal (⧅). If there is a barrier at the left of a ■ cell then the ■ cell becomes a right-moving signal (▨ or ⧄ that we will fix in the following paragraph). If the □ or ■ state sees that there is a problem (like its neighbor saying there is a signal on its right while its state means there is a signal on its left), it changes into a barrier (◫). The right-moving signal (▨ and ⧄) moves to the right and the left-moving signal (⧅) moves to the left. However, each of them bounces and changes direction when it meets a barrier.

If we had only these rules, we would have a stable CA. Now consider the right-moving signals to carry a parity bit: ▨ means there is an even number of □ on its left before having a barrier (◫) and ⧄ means this number is odd. When a left-moving signal (⧅) meets a barrier (or when a right-moving signal is created from a ■ cell), it bounces into an even right-moving signal (▨). When moving to the right, the right-moving signals change their parity bit: ▨ becomes ⧄ and ⧄ becomes ▨. This means that, when it has already bounced on a barrier on its left, a right-moving signal has an even number of □ cells on its left before having a barrier (◫) if and only if it is an even right-moving signal (▨) and an odd number of □ iff it is ⧄.

We add another special case that we will use in Section 4: when an odd right-moving signal (⧄) meets a barrier it does not bounce immediately but rather becomes a □ cell; it does not change much the dynamics –a left-moving signal will be recreated at the next iteration– but this allows us not to have a signal between two barriers in the limit set if and only if the barriers are separated by an even number of

□. If we do not add this condition, the limit set we obtain still has property $(\mathbf{H})$ but we do not know if it can be the limit set of a stable CA.

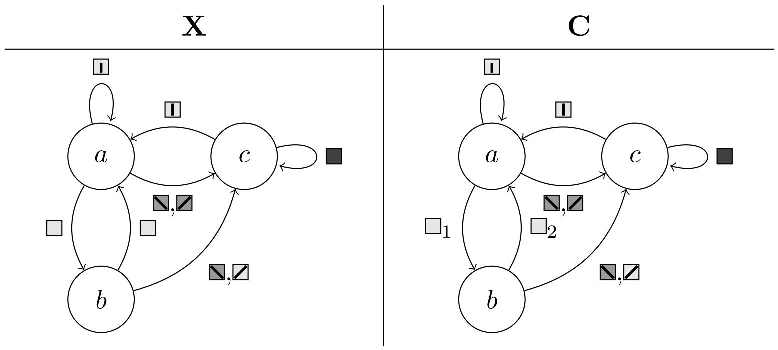


Figure 2. The limit set $\mathbf{X}$ and an SFT cover $\mathbf{C}$ with equal entropy.

From now on, $\mathbf{X}$ will denote the sofic subshift defined over the alphabet Σ represented by the graph in the left part of Figure 2.

Proposition 3.1. U is an unstable CA and its limit set is $\mathbf{X}$: a subshift with property $(\mathbf{H})$.

Proof:
First, let us prove that $\mathbf{X}$ is the limit set of U: following the description of U, it is clear that we have at most one signal between two barriers. If there is no signal, this means that there is an even number of □ cells: if there were an odd number of cells then a signal would have been created and would bounce without ever disappearing. Reading from left to right, after any number of □ cells we can always have a left-moving signal ⧅, then some ■ cells and finally the barrier. We cannot have an even right-moving signal ⧄ after an odd number of □ cells: since our configuration should be in the limit set of U, the signal should have bounced from the wall on its left and be an odd right-moving signal ⧄. The same reasonning applies for odd right-moving signals ⧄ and an even number of □ cells. After a right-moving signal we can always have any number of ■ cells and finally the barrier.

If we have a configuration with only one barrier: on its right it must follow the parity rules for the right-moving signals because the signal should have bounced on the barrier. If we have no barrier, the configuration has at most one signal with □ cells on its left and ■ cells on its right.

It should now be clear that U is unstable: by starting from a pattern ◫⧄ with n cells ■ on its right and then a barrier, we will reach the limit set after at least n steps: when the right-moving signal will hit the barrier and bounce as a left-moving signal.

It is also clear from its description that $\mathbf{X}$ is sofic. $\mathbf{X}$ is mixing because its graph on Figure 2 is strongly connected and has a cycle of length one. ◫ is a magic word, and the corresponding uniform configuration is receptive. □

4. Stable CA

We want to construct a stable CA with limit set $\mathbf{X}$. Subshift $\mathbf{X}$ has property $(\mathbf{H})$ and hence verifies the necessary conditions described in [13] but it is not an AFT and thus does not fall under the scope of the sufficient conditions described. However, we can obtain another sufficient condition that will help us to construct our stable CA.

Lemma 4.1. If a subshift $\mathbf{X}$ is weakly conjugate to a subshift $\mathbf{S}$ that is the limit set of a stable CA then $\mathbf{X}$ is also the limit set of a stable CA.

Proof:
Let Δ be a stable CA with limit set $\mathbf{S} \subseteq \Gamma^{\mathbb{Z}}$. By considering Δ^N instead of Δ, we may assume that $\Delta(\Gamma^{\mathbb{Z}}) = \Delta(\mathbf{S}) = \mathbf{S}$. Let φ and π be block maps such that $\varphi(\mathbf{X}) = \mathbf{S}$ and $\pi(\mathbf{S}) = \mathbf{X}$. If $\mathbf{X} \subset \Sigma^{\mathbb{Z}}$, then φ can actually be extended into $\varphi : \Sigma^{\mathbb{Z}} \to \Gamma^{\mathbb{Z}}$.

We claim that $\pi \circ \Delta \circ \varphi$ is a stable CA with limit set $\mathbf{X}$. Indeed, $\Delta \circ \varphi(\Sigma^{\mathbb{Z}})$ can be written $\Delta(\varphi(\Sigma^{\mathbb{Z}}))$. This includes $\Delta(\varphi(\mathbf{X})) = \Delta(\mathbf{S}) = \mathbf{S}$. Conversely, $\Delta(\varphi(\Sigma^{\mathbb{Z}}))$ is included in $\Delta(\Gamma^{\mathbb{Z}}) = \mathbf{S}$. Putting things together, we obtain that $\Delta(\varphi(\Sigma^{\mathbb{Z}})) = \mathbf{S}$, and $\pi \circ \Delta \circ \varphi(\Sigma^{\mathbb{Z}}) = \pi(\mathbf{S}) = \mathbf{X}$. Moreover, $\pi \circ \Delta \circ \varphi(\mathbf{X}) = \pi \circ \Delta(\mathbf{S}) = \pi(\mathbf{S}) = \mathbf{X}$, which completes the proof. □

Corollary 4.1. If a subshift $\mathbf{X}$ has property $(\mathbf{H})$ and is weakly conjugate to an SFT, then it is the limit set of a stable CA.

Proof:
As a factor of $\mathbf{X}$, such an SFT must also satisfy $(\mathbf{H})$, and by [13, Theorem 3.2], it is the limit set of a stable CA. The claim now follows from Lemma 4.1. □

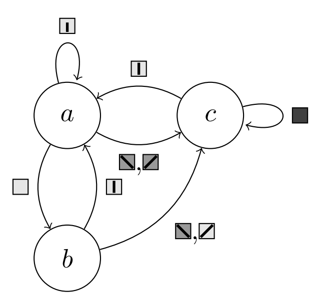
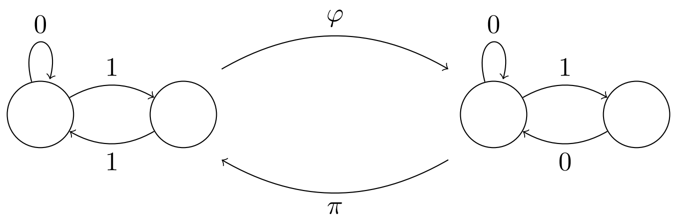


Figure 3. The SFT $\mathbf{S}$.

From now on, $\mathbf{S}$ will denote the SFT depicted on Figure 3. In the following we show that $\mathbf{S}$ and $\mathbf{X}$ are weakly conjugate, which implies, using Corollary 4.1, that there exists a stable CA with limit set $\mathbf{X}$.

The basic idea is simple: we want to check the parity of the □s and we are allowed to use a weak conjugacy to an SFT. It is well known (see *e.g.*, [12]) that the even subshift (containing all configurations on alphabet $\{0, 1\}$ that have an even number of 1s between any two 0s) is not an SFT, but is weakly conjugate to the golden mean subshift (containing all configurations on alphabet $\{0, 1\}$, that never have two consecutive 1s), which is an SFT. These subshifts and the block maps are depicted on Figure 4.

The case of our subshifts is very similar: we forbid two consecutive □s but allow one □ between two barriers: while ◫ plays the role of 0 and □ the role of 1, $\mathbf{X}$ plays the role of the even subshift and $\mathbf{S}$ plays the role of the golden mean subshift. The block map from $\mathbf{S}$ to $\mathbf{X}$ can be constructed by analogy with



	π	φ
000	0	0
001	0	0
010	1	1
011	1	1
100	1	1
101	1	0
110	1	0
111	1	0

Figure 4. The even subshift, the golden mean subshift and their block maps.

the one from the golden mean subshift to the even subshift. The idea is that the patterns □◫ will be used for constructing arbitrary large patterns of □s of even size.

Definition 4.1. π is the block map with neighborhood $\{-1, 0, 1\}$, equal to the identity map except when the following rule matches:

□◫○ → □

Lemma 4.2. $\pi(\mathbf{S}) = \mathbf{X}$.

Proof:
By looking at the graph of $\mathbf{S}$ on Figure 3, it is clear that the only way to obtain □◫ is to follow the aba path. Conversely, the edge labeled ◫ from b to a is always preceded by □. By applying π, □◫ becomes □□, so the path from b to a becomes labeled by □, and we get the same graph as $\mathbf{X}$, depicted in Figure 2. □

Note that $\mathbf{X}$ and $\mathbf{C}$ from Figure 2, and also $\mathbf{S}$ from Figure 3 have the same entropy, which is a necessary condition for the following results. In order to match the conditions of Corollary 4.1, we have to build a block map from $\mathbf{X}$ onto $\mathbf{S}$. This can be derived from the block map from the even subshift to the golden mean subshift.

Definition 4.2. φ is the block map defined by neighborhood $\{-1, 0, 1\}$ and the following local rule, where the first matching rule is used and, if none matches, the identity map is applied:

□◫◫	→	□	□◫⧅	→	□
□□◫	→	◫	□□□	→	◫
□□⧄	→	◫	□□⧅	→	◫

We know that the image of a sofic subshift by a block map can be computed from its graph and the local rule defining the block map [12] but in our case the graphs obtained tend to be rather big. The tedious part in proving that the image of a block map is a given subshift is to prove that the map is onto. Fortunately, in the case of irreducible sofic subshifts, it suffices to prove that a block map is right-closing.

Definition 4.3. A block map $\varphi : \mathbf{X} \to \mathbf{Y}$ is said to be *right-closing* if, for any $x, y \in \mathbf{X}$ such that $x_{|-\mathbb{N}} = y_{|-\mathbb{N}}$, $\varphi(x) = \varphi(y)$ if and only if $x = y$. In other words, φ does not collapse any two left-asymptotic words.

Proposition 4.1. ([12])
If a block map φ defined over an irreducible sofic subshift $\mathbf{X}$ is right-closing then $h(\varphi(\mathbf{X})) = h(\mathbf{X})$.

Lemma 4.3. ([12, Corollary 4.4.9])
A proper subshift of an irreducible sofic subshift has a strictly lower entropy.

Lemma 4.4. $\varphi(\mathbf{X}) = \mathbf{S}$

Proof:
We first prove that $\varphi(\mathbf{X}) \subseteq \mathbf{S}$ and then that φ is onto.

$\varphi(\mathbf{X}) \subseteq \mathbf{S}$**:** Since φ is the identity map for ■, ⧅, ⧄ and ⧄ cells that obey the same rules for $\mathbf{X}$ and $\mathbf{S}$, we only have to prove that no □□ can appear in $\varphi(\mathbf{X})$. Let us do a case by case analysis of how to obtain a □ cell by φ:

- □◫◫: in $\mathbf{X}$, there must be a □ on the left and (abusively written) $\varphi($□□◫◫$) =$ ○◫□◫, hence this rule cannot be used to obtain two consecutive □ cells.

- □◫⧅: again, in $\mathbf{X}$, there must be a □ on the left and we can apply the same argument.

- ○□⧄: the ○ cell can be either □ or ◫; if it is ◫ then $\varphi($◫□⧄$) =$ ◫□⧄; if it is □ then there must be another □ on the left since the number of □ between a ◫ and a ⧄ must be odd and $\varphi($□□□⧄$) =$ ○◫□⧄. Again, this rule cannot be used to obtain two consecutive □ cells.

- ◫□○: the ○ cell can be either □, ⧅ or ⧄; the case of ⧄ has just been done; $\varphi($◫□⧅$) =$ ◫□⧅; if it is □, the only way that the rightmost □ does not become a ◫ by φ is that there is a ⧄ cell on its right but this is forbidden in $\mathbf{X}$.

φ **is onto:** Let $\Psi : \mathbf{C} \to \mathbf{X}$ be the canonical projection: $\Psi(\square_1) = \Psi(\square_2) = \square$ and the identity over the other symbols. Since $\varphi(\mathbf{X}) \subseteq \mathbf{S}$ and $\mathbf{C}$ and $\mathbf{S}$ are irreducible SFTs of equal entropy, $\varphi \circ \Psi : \mathbf{C} \to \mathbf{S}$ is onto if and only if $h(\varphi \circ \Psi(\mathbf{C})) = h(\mathbf{C})$ by Lemma 4.3 (or [12, Corollary 4.4.9]). In this case, φ is also onto. By Proposition 4.1, it suffices to prove that $\varphi \circ \Psi$ is right-closing.

Let c_1 and c_2 be two different left-asymptotic configurations of $\mathbf{C}$. The only way to change a symbol by φ is on a ◫ or $\square$ cell with a $\square$ cell on their left. Therefore, the leftmost position where c_1 and c_2 differ starts with $\square_2\square_1$ for c_1 and $\square_2$◫ for c_2 (or the other way around). There are now two cases:

- If the ◫ in c_2 is changed into $\square$ while $\square$ in c_1 remains unchanged, then the next symbol in c_1 can only be ⧄. But then also the next symbol in c_2 should be ⧄ since ⧄ never changes. We should have the following:

  $$\begin{array}{ccccccc} c_1 & = & \dots & \square_2 & \square_1 & ⧄ & \dots \\ c_2 & = & \dots & \square_2 & ◫ & ⧄ & \dots \\ \varphi\circ\Psi & \to & & \dots & \square & ⧄ & \dots \end{array}$$

  But ◫⧄ is forbidden in $\mathbf{C}$.

- If the $\square_1$ in c_1 is changed into ◫ while ◫ in c_2 remains unchanged then the next symbol in c_1 must be $\square_2$ or ▧, and the next symbol in c_2 must be $\square_1$ or ▨. Since ▧ and ▨ cells remain unchanged by φ, the next symbol in c_1 cannot be ▧, and the next symbol in c_2 cannot be ▨. They continue as follows:

  $$\begin{array}{ccccccc} c_1 & = & \dots & \square_2 & \square_1 & \square_2 & \dots \\ c_2 & = & \dots & \square_2 & ◫ & \square_1 & \dots \end{array}$$

  The rightmost $\square_1$ in c_2 is mapped to $\square$; the only case where the rightmost $\square_2$ in c_1 is mapped to $\square$ is when it is followed by a ⧄. They should therefore continue like this:

  $$\begin{array}{cccccccc} c_1 & = & \dots & \square_2 & \square_1 & \square_2 & ? & \dots \\ c_2 & = & \dots & \square_2 & ◫ & \square_1 & ⧄ & \dots \end{array}$$

  The ⧄ cells never change and therefore c_1 should continue with a ⧄ but in $\mathbf{C}$, a $\square_2$ cannot be followed by a ⧄.

We conclude that $\varphi \circ \Psi$ is right-closing, completing the proof. $\square$

Proposition 4.2. $\mathbf{X}$ is the limit set of a stable CA.

Proof:
By Lemma 4.2 and Lemma 4.4, $\mathbf{X}$ is weakly conjugate to $\mathbf{S}$. Moreover, by Proposition 3.1, $\mathbf{X}$ has property $(\mathbf{H})$. The claim now follows from Corollary 4.1. $\square$

Figure 5 shows space-time diagrams of the stable CA obtained with Proposition 4.2.

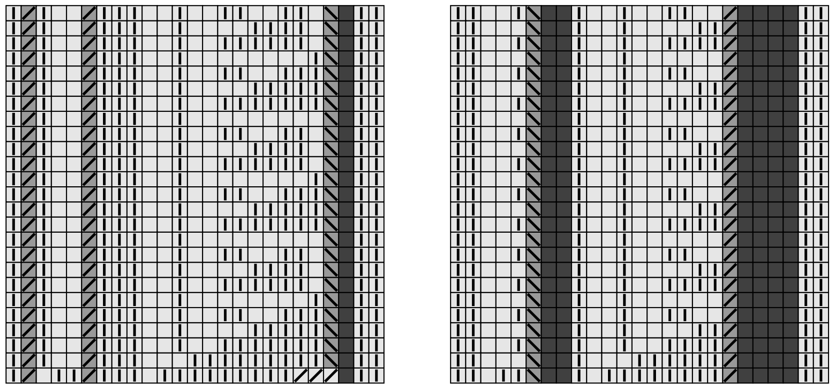


Figure 5. The stable CA obtained from Proposition 4.2.

5. Conclusions, conjectures and open problems

In this paper we gave an example of a subshift that is the limit set of a stable and an unstable CA, answering a question that has been left open since 15 years [13]. Unfortunately, we have not been able to come to a characterization of subshifts that are limit sets of stable CAs. This problem deserves an entire paragraph in M. Boyle's compilation of open problems in symbolic dynamics [3, § 16]. Our proof of Proposition 4.2 proves that the subshift is weakly conjugate to its Shannon cover which is a mixing SFT with property $(\mathbf{H})$; A. Maass's proof that AFTs with property $(\mathbf{H})$ are limit sets of stable CAs is actually based on the fact, due to J. Ashley, that these subshifts are weakly conjugate to their Shannon covers [1] (but [13] contains counter-examples for the converse). Based on these remarks, we conjecture the following.

Conjecture 1. The limit set of any stable CA is weakly conjugate to an SFT.

Though weak conjugacy is well understood among SFTs, being equivalent to shift equivalence [12], it was never really studied for itself, to our knowledge. It would be interesting to understand which subshifts are weakly conjugate to SFTs.

The main problem remains to have a nice characterization of limit sets, if not of all CAs, at least of stable CAs. Since they are all sofic, hence finitely describable by their graph, we can ask the following question.

Open problem 1. Is it decidable whether a sofic subshift is the limit set of some stable CA?

The unstable case is probably known even worse. A. Maass has proven that some class intermediary between SFTs and AFTs, namely the near-Markov subshifts, cannot be limit sets of any unstable CA; it is not known whether this can be generalized to AFTs –note that our example is not AFT.

Finally, it is to be noted that very little is known on the corresponding problematic for CAs of higher dimension. Even a simple sufficient condition for SFTs to be the limit set of a bidimensional CA seems much more difficult to achieve.

References

[1] Ashley, J.: An extension theorem for closing maps of shifts of finite type, *Transactions of the American Mathematical Society*, **336**, 1993, 389–420.

[2] Boyle, M.: Lower entropy factors of sofic systems, *Ergodic theory and dynamical systems*, **4**, 1984, 541–557.

[3] Boyle, M.: Open problems in symbolic dynamics, *Contemporary Mathematics*, **469**, 2008, 69–118.

[4] Culik II, K., Pachl, J., Yu, S.: On the limit sets of cellular automata, *SIAM Journal of Computing*, **18(4)**, 1989, 831.

[5] Formenti, E., Kůrka, P.: Subshift attractors of cellular automata, *Nonlinearity*, **20**(1), 2007, 105–117.

[6] Hedlund, G. A.: Endomorphisms and Automorphisms of the Shift Dynamical Systems., *Mathematical Systems Theory*, **3**(4), 1969, 320–375.

[7] Hurd, L. P.: Formal Language Characterizations of Cellular Automaton Limit Sets, *Complex Systems*, **1**, 1987, 69–80.

[8] Hurd, L. P.: Nonrecursive Cellular Automata Invariant Sets, *Complex Systems*, **4**(2), April 1990, 131–138.

[9] Hurd, L. P.: Recursive Cellular Automata Invariant Sets, *Complex Systems*, **4**(2), April 1990, 119–129.

[10] Kari, J.: Rice's theorem for the limit set of cellular automata, *Theoretical Computer Science*, **127**(2), 1994, 229–254.

[11] di Lena, P., Margara, L.: Undecidable Properties of Limit Set Dynamics of Cellular Automata, *STACS* (S. Albers, J.-Y. Marion, Eds.), 3, Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, Germany, 2009, ISBN 978-3-939897-09-5, 337–347.

[12] Lind, D. A., Marcus, B.: *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, New York, NY, USA, 1995, ISBN 0521551242.

[13] Maass, A.: On the sofic limit sets of cellular automata, *Ergodic Theory and Dynamical Systems*, **15**, 1995, 663–684.